# Determinants

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# Outline

- Basic properties
- Derived properties
- Computation
- Application

## NOTATION

Let $\boldsymbol{A}$ be a matrix. We use the following notation for certain $\boldsymbol{A}$-related matrices.

- $\boldsymbol{A}_{i \leftrightarrow j}$: exchanging row $i$ and row $j$
- $\boldsymbol{A}_{\boldsymbol{a}_{i:}=\boldsymbol{b}^T}$ or $\boldsymbol{A}_{\boldsymbol{a}_{i:} \leftarrow \boldsymbol{b}^T}$: setting or replacing row $i$ with $\boldsymbol{b}^T$
- $\boldsymbol{A}_{\boldsymbol{a}_j=\boldsymbol{b}}$ or $\boldsymbol{A}_{\boldsymbol{a}_j \leftarrow \boldsymbol{b}}$: setting or replacing column $j$ with $\boldsymbol{b}$
- $\boldsymbol{A}_{\boldsymbol{a}_{i:} \leftarrow \boldsymbol{a}_{i:} - m\boldsymbol{a}_{j:}}$: row operation ($e_{ij} = -m$)
- $\boldsymbol{M}_{ij}$: removing row $i$ and column $j$

## Basic properties (BP)

## BP1: IDENTITY MATRIX

The determinant of an **identity matrix** is 1.

$$|\boldsymbol{I}_2| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$|\boldsymbol{I}_3| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

## BP2: Exchange of adjacent rows

The determinant of a matrix changes its sign with an **exchange of adjacent rows**

$$|\boldsymbol{A}_{i\leftrightarrow i+1}| = -|\boldsymbol{A}|$$

$$|\boldsymbol{A}_{i\leftrightarrow i-1}| = -|\boldsymbol{A}|$$

$$\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

$$\begin{vmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ -1 & 0 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ -1 & 0 & 1 \end{vmatrix}$$

## BP3: Linearity in the first row

The determinant of a matrix is **linear in the first row**

$$|\boldsymbol{A}_{\boldsymbol{a}_{1:}=\alpha\boldsymbol{b}^T+\beta\boldsymbol{c}^T}| = \alpha|\boldsymbol{A}_{\boldsymbol{a}_{1:}=\boldsymbol{b}^T}| + \beta|\boldsymbol{A}_{\boldsymbol{a}_{1:}=\boldsymbol{c}^T}|$$

$$\begin{vmatrix} 3 & 7 \\ 3 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 3 & 2 \end{vmatrix} + \begin{vmatrix} 0 & 7 \\ 3 & 2 \end{vmatrix}$$

$$\begin{vmatrix} 3 & 7 & 1 \\ 3 & 2 & 1 \\ 1 & -1 & 2 \end{vmatrix} + \begin{vmatrix} -2 & -3 & -3 \\ 3 & 2 & 1 \\ 1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 4 & -2 \\ 3 & 2 & 1 \\ 1 & -1 & 2 \end{vmatrix}$$

## Corollary (exchange of rows)

*The determinant of a matrix changes its sign with an exchange of any two rows*

$$|\boldsymbol{A}_{i\leftrightarrow j}| = -|\boldsymbol{A}|,\ i \neq j$$

$$\begin{vmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ -1 & 0 & 1 \end{vmatrix} = (-1)\begin{vmatrix} -1 & 0 & 1 \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{vmatrix} = (-1)^2\begin{vmatrix} -1 & 0 & 1 \\ 3 & 4 & 5 \\ 1 & 2 & 3 \end{vmatrix}$$

**Note.** Exchange of non-adjacent rows is equivalent to an odd number of exchanges of adjacent rows.

## Corollary (permutation matrix)

*The determinant of a permutation matrix is* $\pm 1$.

$$\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 1, \quad \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{vmatrix} = -1$$

**Note.** A permutation matrix is related to an identity matrix of the same order by row exchanges.

## Corollary (linearity in any row)

*The determinant of a matrix is* **linear in any row**. *That is*

$$|\boldsymbol{A}_{\boldsymbol{a}_{i:}=\alpha\boldsymbol{b}^T+\beta\boldsymbol{c}^T}| = \alpha|\boldsymbol{A}_{\boldsymbol{a}_{i:}=\boldsymbol{b}^T}| + \beta|\boldsymbol{A}_{\boldsymbol{a}_{i:}=\boldsymbol{c}^T}|$$

$$\begin{vmatrix} \boldsymbol{a}_{1:} \\ \cdot \\ \alpha\boldsymbol{b}^T + \beta\boldsymbol{c}^T \\ \cdot \end{vmatrix} = -\begin{vmatrix} \alpha\boldsymbol{b}^T + \beta\boldsymbol{c}^T \\ \cdot \\ \boldsymbol{a}_{1:} \\ \cdot \end{vmatrix} = -\alpha\begin{vmatrix} \boldsymbol{b}^T \\ \cdot \\ \boldsymbol{a}_{1:} \\ \cdot \end{vmatrix} - \beta\begin{vmatrix} \boldsymbol{c}^T \\ \cdot \\ \boldsymbol{a}_{1:} \\ \cdot \end{vmatrix}$$

$$= \alpha\begin{vmatrix} \boldsymbol{a}_{1:} \\ \cdot \\ \boldsymbol{b}^T \\ \cdot \end{vmatrix} + \beta\begin{vmatrix} \boldsymbol{a}_{1:} \\ \cdot \\ \boldsymbol{c}^T \\ \cdot \end{vmatrix}$$

## Derived properties (DP)

## DP1: equal rows

The determinant of a matrix with two equal rows is 0.

Let $\boldsymbol{A}$ be a matrix with $\boldsymbol{a}_{i:} = \boldsymbol{a}_{j:}$. Consider the determinant of $\boldsymbol{A}_{i\leftrightarrow j}$. Note

$$\overbrace{|\boldsymbol{A}_{i\leftrightarrow j}| = -|\boldsymbol{A}|}^{\text{row exchange}} \text{ and } \overbrace{|\boldsymbol{A}_{i\leftrightarrow j}| = |\boldsymbol{A}|}^{\boldsymbol{A}_{i\leftrightarrow j}=\boldsymbol{A}}$$

So $-|\boldsymbol{A}| = |\boldsymbol{A}|$. Hence

$$|\boldsymbol{A}| = 0$$

## DP2: ROW OPERATION

The determinant of a matrix is invariant with respect to row operation.

Consider the determinant of $\boldsymbol{A}_{\boldsymbol{a}_{i:} \leftarrow \boldsymbol{a}_{i:} - m\boldsymbol{a}_{j:}}$.

$$
\begin{aligned}
\left|\boldsymbol{A}_{\boldsymbol{a}_{i:} \leftarrow \boldsymbol{a}_{i:} - m\boldsymbol{a}_{j:}}\right| &= \overbrace{\left|\boldsymbol{A}_{\boldsymbol{a}_{i:} \leftarrow \boldsymbol{a}_{i:}}\right| - \left|\boldsymbol{A}_{\boldsymbol{a}_{i:} \leftarrow m\boldsymbol{a}_{j:}}\right|}^{\text{linearity}} \\
&= |\boldsymbol{A}| - m \overbrace{\left|\boldsymbol{A}_{\boldsymbol{a}_{i:} \leftarrow \boldsymbol{a}_{j:}}\right|}^{\text{equal rows}} \\
&= |\boldsymbol{A}|
\end{aligned}
$$

## DP3: zero row

The determinant of a matrix with a zero row is 0.

Let $\boldsymbol{A}$ be a matrix with $\boldsymbol{a}_{i:} = \mathbf{0}^T$. Consider the determinant of $\boldsymbol{A}_{\boldsymbol{a}_{i:} \leftarrow \boldsymbol{a}_{i:} + \boldsymbol{a}_{j:}}$ where $j \neq i$.

$$|\boldsymbol{A}| = \overbrace{|\boldsymbol{A}_{\boldsymbol{a}_{i:} \leftarrow \boldsymbol{a}_{i:} + \boldsymbol{a}_{j:}}|}^{\text{row operation}} = |\boldsymbol{A}_{\boldsymbol{a}_{i:} \leftarrow \mathbf{0}^T + \boldsymbol{a}_{j:}}| = \overbrace{|\boldsymbol{A}_{\boldsymbol{a}_{i:} \leftarrow \boldsymbol{a}_{j:}}|}^{\text{equal rows}} = 0$$

## Example

- equal rows

$$\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0$$

- row operation

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix}$$

- zero row

$$\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 0$$

## DP4: triangular matrix

The determinant of a lower-triangular matrix is the product of the diagonal elements.

If every diagonal element of $\boldsymbol{A}$ is non-zero, we can use row operations to eliminate the non-diagonal elements. Thus

$$\boldsymbol{A} \xrightarrow{\text{row operations}} \boldsymbol{D} \Rightarrow |\boldsymbol{A}| = |\boldsymbol{D}| = a_{11} \ldots a_{nn} |\boldsymbol{I}_n| = a_{11} \ldots a_{nn}$$

Otherwise, let $a_{kk}$ be the topmost zero on the diagonal of $\boldsymbol{A}$. We can use row operations to convert $\boldsymbol{A}$ to $\boldsymbol{A}'$, eliminating the elements to the left of $a_{kk}$. Then, since the elements of $\boldsymbol{A}'$ on row $k$ are all zeros, we have

$$|\boldsymbol{A}| = |\boldsymbol{A}'| = 0 = a_{11} \ldots a_{nn}$$

## DP5: singular matrix

The determinant of a non-singular matrix cannot be 0. The determinant of a singular matrix is 0.

Suppose $\boldsymbol{A}$ is non-singular. Through row operations and row exchanges, $\boldsymbol{A}$ is converted to an upper-triangular $\boldsymbol{U}$ with full pivots. Hence

$$|\boldsymbol{A}| = \pm|\boldsymbol{U}| = \pm\prod_i u_{ii} \neq 0$$

Suppose $\boldsymbol{A}$ is singular. We still have $\boldsymbol{A}$ converted to $\boldsymbol{U}$ with at least one zero row. Hence

$$|\boldsymbol{A}| = \pm|\boldsymbol{U}| = 0$$

## DP6: MULTIPLICATION

The determinant of the product of two matrices equals the product of the determinants of the matrices. That is

$$|\boldsymbol{AB}| = |\boldsymbol{A}||\boldsymbol{B}|$$

If $\boldsymbol{B}$ is singular, then $\boldsymbol{AB}$ is also singular and

$$|\boldsymbol{AB}| = 0 = |\boldsymbol{A}||\boldsymbol{B}|$$

If $\boldsymbol{B}$ is non-singular, we show that the BPs of determinant are satisfied by

$$f(\boldsymbol{A}) = \frac{|\boldsymbol{AB}|}{|\boldsymbol{B}|}$$

So $f(\boldsymbol{A}) = |\boldsymbol{A}|$ and

$$|\boldsymbol{AB}| = |\boldsymbol{A}||\boldsymbol{B}|$$

## DETAILS OF DP6.

- BP1

$$f(\boldsymbol{I}) = \frac{|\boldsymbol{IB}|}{|\boldsymbol{B}|} = \frac{|\boldsymbol{B}|}{|\boldsymbol{B}|} = 1$$

- BP2

$$\begin{aligned} & \boldsymbol{A}_{i\leftrightarrow i+1}\boldsymbol{B} = (\boldsymbol{AB})_{i\leftrightarrow i+1} \\ \Rightarrow\ & |\boldsymbol{A}_{i\leftrightarrow i+1}\boldsymbol{B}| = |(\boldsymbol{AB})_{i\leftrightarrow i+1}| = -|\boldsymbol{AB}| \\ \Rightarrow\ & f(\boldsymbol{A}_{i\leftrightarrow i+1}) = -f(\boldsymbol{A}) \end{aligned}$$

- BP3

$$\begin{aligned} & \left(\alpha\boldsymbol{c}^T + \beta\boldsymbol{d}^T\right)\boldsymbol{B} = \alpha\boldsymbol{c}^T\boldsymbol{B} + \beta\boldsymbol{d}^T\boldsymbol{B} \\ \Rightarrow\ & |\boldsymbol{A}_{\boldsymbol{a}_{1:}=\alpha\boldsymbol{c}^T+\beta\boldsymbol{d}^T}\boldsymbol{B}| = \alpha|\boldsymbol{A}_{\boldsymbol{a}_{1:}=\boldsymbol{c}^T}\boldsymbol{B}| + \beta|\boldsymbol{A}_{\boldsymbol{a}_{1:}=\boldsymbol{d}^T}\boldsymbol{B}| \\ \Rightarrow\ & f(\boldsymbol{A}_{\boldsymbol{a}_{1:}=\alpha\boldsymbol{c}^T+\beta\boldsymbol{d}^T}) = \alpha f(\boldsymbol{A}_{\boldsymbol{a}_{1:}=\boldsymbol{c}^T}) + \beta f(\boldsymbol{A}_{\boldsymbol{a}_{1:}=\boldsymbol{d}^T}) \end{aligned}$$

Hence $|\boldsymbol{AB}| = |\boldsymbol{A}||\boldsymbol{B}|$. □

## DP7: Transpose

The determinant of a matrix is invariant to matrix transpose. That is

$$|\boldsymbol{A}| = |\boldsymbol{A}^T|$$

If $\boldsymbol{A}$ is singular, then $\boldsymbol{A}^T$ is singular and $|\boldsymbol{A}| = |\boldsymbol{A}^T| = 0$. Otherwise, from the LDU decomposition $\boldsymbol{PA} = \boldsymbol{LDU}$

$$\begin{aligned} |\boldsymbol{P}||\boldsymbol{A}| &= |\boldsymbol{L}|\,|\boldsymbol{D}|\,|\boldsymbol{U}| = \left|\boldsymbol{U}^T\right|\left|\boldsymbol{D}^T\right|\left|\boldsymbol{L}^T\right| \\ &= \left|\boldsymbol{U}^T\boldsymbol{D}^T\boldsymbol{L}^T\right| = \left|\boldsymbol{A}^T\boldsymbol{P}^T\right| = \left|\boldsymbol{A}^T\right|\left|\boldsymbol{P}^T\right| \end{aligned}$$

Since $\boldsymbol{P}^T\boldsymbol{P} = \boldsymbol{I}$ and $|\boldsymbol{P}| = \pm 1$, we have

$$\left|\boldsymbol{P}^T\boldsymbol{P}\right| = 1 = \left|\boldsymbol{P}^T\right|\left|\boldsymbol{P}\right| \;\Rightarrow\; |\boldsymbol{P}| = \left|\boldsymbol{P}^T\right|$$

Hence

$$|\boldsymbol{A}| = \left|\boldsymbol{A}^T\right|$$

## Example

- triangular matrix

$$\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} = 1 \cdot 3$$

- singular matrix

$$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0 \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \text{ singular}$$

- matrix multiplication

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix} = \begin{vmatrix} 8 & 5 \\ 20 & 13 \end{vmatrix}$$

- matrix transpose

$$\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

## Computation of Determinant

**Theorem (determinant as product of pivots)**

*Let $\boldsymbol{A}$ be a non-singular matrix. $|\boldsymbol{A}|$ equals the product of the pivots of $\boldsymbol{A}$, possibly apart from a sign.*

Let the LDU decomposition of $\boldsymbol{A}$ be $\boldsymbol{PA} = \boldsymbol{LDU}$. Then

$$|\boldsymbol{P}||\boldsymbol{A}| = |\boldsymbol{L}||\boldsymbol{D}||\boldsymbol{U}|$$

$\boldsymbol{L}$ and $\boldsymbol{U}$ are unit-triangular, so $|\boldsymbol{L}| = |\boldsymbol{U}| = 1$. $\boldsymbol{P}$ is a permutation matrix, so $|\boldsymbol{P}| = \pm 1$. Thus

$$|\boldsymbol{A}| = \pm|\boldsymbol{D}| = \pm \prod_i d_i$$

## Example (determinant as product of pivots)

$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \cdot & \\ & & \cdot & \cdot & -1 \\ & & & -1 & 2 \end{bmatrix} = L \begin{bmatrix} 2 & & & & \\ & \frac{3}{2} & & & \\ & & \frac{4}{3} & & \\ & & & \cdot & \\ & & & & \frac{n+1}{n} \end{bmatrix} U$$

The pivots of $A$ are

$$2, \frac{3}{2}, \frac{4}{3}, \ldots, \frac{n+1}{n}$$

Hence

$$|A| = 2\left(\frac{3}{2}\right)\left(\frac{4}{3}\right)\ldots\left(\frac{n+1}{n}\right) = n+1$$

## Definition (parity of permutation)

Let $(j_1 : j_n)$ be a permutation of $(1 : n)$.

- There are $n!/2!$ pairs
- A pair $(j_k, j_l)$ with $k < l$ is reversed if $j_k > j_l$
- A permutation has odd parity if the total count of reversed pairs is odd
- A permutation has even parity if the total count of reversed pairs is even

For example, consider $(321)$ as a permutation of $(123)$.

- $3!/2! = 3$ pairs
- Pairs $(3, 2)$, $(3, 1)$, and $(2, 1)$ are reversed pairs
- Other pairs are not reversed
- $(321)$ is an odd permutation of $(123)$

## Lemma (determinant of permutation matrix)

*Let $\boldsymbol{P}(j_1 : j_n)$ be the permutation matrix where $(j_1 : j_n)$ is a permutation of $(1 : n)$ and $p_{ij_i} = 1$ for $i = 1, \dots, n$. Let $z(j_1 : j_n)$ be the number of reversed pairs of $(j_1 : j_n)$. Then*

$$|\boldsymbol{P}(j_1 : j_n)| = (-1)^{z(j_1 : j_n)}$$

For example

$$\boldsymbol{P}(321) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and

$$|\boldsymbol{P}(321)| = (-1)^3 = -1$$

**Proof.**

Convert $\boldsymbol{P}(j_1 : j_n)$ to $\boldsymbol{I}$ through exchanges of adjacent rows.

- Exchange adjacent rows if the 1 in the lower row is to the left of the 1 in the upper row
- Equivalent to converting $(j_1 : j_n)$ to $(1 : n)$ by swapping adjacent reversed pairs
- Each swap of adjacent reversed pair reduces the number of reversed pairs by 1, so it requires $z(j_1 : j_n)$ swaps to convert $(j_1 : j_n)$ to $(1 : n)$
- Thus it also requires $z(j_1 : j_n)$ exchanges of adjacent rows to convert $\boldsymbol{P}(j_1 : j_n)$ to $\boldsymbol{I}$

Hence

$$|\boldsymbol{P}(j_1 : j_n)| = (-1)^{z(j_1:j_n)}|\boldsymbol{I}| = (-1)^{z(j_1:j_n)}$$

□

## Theorem (determinant as sum over permutation)

*Let $\boldsymbol{A}$ be a matrix of order $n \times n$.*

$$|\boldsymbol{A}| = \sum_{\textit{even } (j_1 : j_n)} a_{1j_1} \dots a_{nj_n} - \sum_{\textit{odd } (j_1 : j_n)} a_{1j_1} \dots a_{nj_n}$$

**Proof***. $|\boldsymbol{A}|$ can be expressed as the sum of $n^n$ terms

$$|\boldsymbol{A}| = \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} a_{1j_1} \dots a_{nj_n} \, |\boldsymbol{Q}(j_1 : j_n)|$$

where $\boldsymbol{Q}(j_1 : j_n)$ is binary with $q_{1j_1} = \cdots = q_{nj_n} = 1$ and 0s elsewhere. Note $|\boldsymbol{Q}(j_1 : j_n)| = |\boldsymbol{P}(j_1 : j_n)| = (-1)^{z(j_1:j_n)}$ if $(j_1 : j_n)$ is a permutation of $(1 : n)$, and 0 otherwise. Thus

$$|\boldsymbol{A}| = \sum_{(j_1:j_n)} a_{1j_1} \dots a_{nj_n} \, |\boldsymbol{P}(j_1 : j_n)| = \sum_{(j_1:j_n)} a_{1j_1} \dots a_{nj_n} \, (-1)^{z(j_1:j_n)}$$

$$= \sum_{\text{even } (j_1:j_n)} a_{1j_1} \dots a_{nj_n} - \sum_{\text{odd } (j_1:j_n)} a_{1j_1} \dots a_{nj_n}$$

## Example (sum over permutation)

$$
\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
= \begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & & \\ & & a_{23} \\ & a_{32} & \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ & & a_{23} \\ a_{31} & & \end{vmatrix}
$$

$$
+ \begin{vmatrix} & a_{12} & \\ a_{21} & & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & & \\ & a_{32} & \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ & a_{22} & \\ a_{31} & & \end{vmatrix}
$$

$$
= (-1)^{z(123)} a_{11}a_{22}a_{33} + (-1)^{z(132)} a_{11}a_{23}a_{32}
$$

$$
+ (-1)^{z(231)} a_{12}a_{23}a_{31} + (-1)^{z(213)} a_{12}a_{21}a_{33}
$$

$$
+ (-1)^{z(312)} a_{13}a_{21}a_{32} + (-1)^{z(321)} a_{13}a_{22}a_{31}
$$

## Definition (cofactors and cofactor matrix)

Let $\boldsymbol{A}$ be a square matrix.

- The cofactor of $a_{ij}$, denoted by $c_{ij}$, is $(-1)^{i+j}|\boldsymbol{M}_{ij}|$, where $\boldsymbol{M}_{ij}$ is the matrix formed by removing row $i$ and column $j$
- The cofactor matrix of $\boldsymbol{A}$ is $\boldsymbol{C} = \{c_{ij}\}$

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$c_{11} = (-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \; c_{32} = (-1)^{3+2}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}, \quad c_{ij} = (-1)^{i+j}|\boldsymbol{M}_{ij}|$$

## Example (cofactor matrix)

For $\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}$, the cofactor matrix is

$$\begin{bmatrix} +\begin{vmatrix}1 & 1\\ 0 & 1\end{vmatrix} & -\begin{vmatrix}0 & 1\\ 0 & 1\end{vmatrix} & +\begin{vmatrix}0 & 1\\ 0 & 0\end{vmatrix}\\ -\begin{vmatrix}1 & 1\\ 0 & 1\end{vmatrix} & +\begin{vmatrix}1 & 1\\ 0 & 1\end{vmatrix} & -\begin{vmatrix}1 & 1\\ 0 & 0\end{vmatrix}\\ +\begin{vmatrix}1 & 1\\ 1 & 1\end{vmatrix} & -\begin{vmatrix}1 & 1\\ 0 & 1\end{vmatrix} & +\begin{vmatrix}1 & 1\\ 0 & 1\end{vmatrix}\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 0 & -1 & 1\end{bmatrix}$$

For $\begin{bmatrix}1 & 0 & -1\\ -1 & 1 & 0\\ 0 & -1 & 1\end{bmatrix}$, the cofactor matrix is $\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$.

## PROPERTIES OF COFACTORS

Let $\boldsymbol{A}$ be a square matrix and $\boldsymbol{C}$ be its cofactor matrix.

- $c_{ij}$ does not depend on $\boldsymbol{a}_{i:}$ or $\boldsymbol{a}_j$
- $\boldsymbol{c}_{i:}$ does not depend on $\boldsymbol{a}_{i:}$ and $\boldsymbol{c}_j$ does not depend on $\boldsymbol{a}_j$
  - $\boldsymbol{A}$ and $\boldsymbol{A}_{\boldsymbol{a}_{i:} \leftarrow \boldsymbol{u}^T}$ have the same cofactors along row $i$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ and } \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ u_1 & u_2 & u_3 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

  - $\boldsymbol{A}$ and $\boldsymbol{A}_{\boldsymbol{a}_j \leftarrow \boldsymbol{v}}$ have the same cofactors along column $j$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ and } \begin{bmatrix} a_{11} & a_{12} & v_1 \\ a_{21} & a_{22} & v_2 \\ a_{31} & a_{32} & v_3 \end{bmatrix}$$

## Theorem (determinant as cofactor expansion)

*Let $\boldsymbol{A}$ be a square matrix.*

$$|\boldsymbol{A}| = a_{11}c_{11} + \cdots + a_{1n}c_{1n}$$
$$= \sum_{j=1}^{n} a_{1j}(-1)^{1+j}|\boldsymbol{M}_{1j}|$$

- This is the cofactor expansion of $|\boldsymbol{A}|$ along row 1
- The formula is consistent with BP3 of matrix determinant
- Cofactor expansion can be applied recursively

**Proof***. $|\boldsymbol{A}| = \sum_{j=1}^{n} |\boldsymbol{K}_j|$, where

$$|\boldsymbol{K}_j| = \begin{vmatrix} 0 & \dots & 0 & a_{1j} & 0 & \dots & 0 \\ a_{21} & \dots & a_{2(j-1)} & a_{2j} & a_{2(j+1)} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{n(j-1)} & a_{nj} & a_{n(j+1)} & \dots & a_{nn} \end{vmatrix}$$

If $a_{1j} = 0$ then $|\boldsymbol{K}_j| = 0 = a_{1j}c_{1j}$. If $a_{1j} \neq 0$, we eliminate the elements below $a_{1j}$ by row operations, and

$$|\boldsymbol{K}_j| = \begin{vmatrix} 0 & \dots & 0 & a_{1j} & 0 & \dots & 0 \\ a_{21} & \dots & a_{2(j-1)} & 0 & a_{2(j+1)} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{n(j-1)} & 0 & a_{n(j+1)} & \dots & a_{nn} \end{vmatrix}$$

With $(j-1)$ exchanges of adjacent columns

$$
\begin{aligned}
|\boldsymbol{K}_j| &= (-1)^{j-1}\begin{vmatrix} a_{1j} & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & a_{21} & \dots & a_{2(j-1)} & a_{2(j+1)} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_{n1} & \dots & a_{n(j-1)} & a_{n(j+1)} & \dots & a_{nn} \end{vmatrix} \\
&= a_{1j}(-1)^{j-1}|\boldsymbol{M}_{1j}| \\
&= a_{1j}(-1)^{j+1}|\boldsymbol{M}_{1j}| \\
&= a_{1j}c_{1j}
\end{aligned}
$$

Thus

$$
|\boldsymbol{A}| = \sum_{j=1}^{n}|\boldsymbol{K}_j| = \sum_{j=1}^{n} a_{1j}c_{1j} = a_{11}c_{11} + a_{12}c_{12} + \cdots + a_{1n}c_{1n}
$$

## Corollary (general cofactor expansion)

*Let $\boldsymbol{A}$ be a square matrix. $|\boldsymbol{A}|$ can be expressed as a cofactor expansion along any row or column.*

- The cofactor expansion of $|\boldsymbol{A}|$ along row $i$ is

$$|\boldsymbol{A}| = a_{i1}c_{i1} + \cdots + a_{in}c_{in}$$

- The cofactor expansion of $|\boldsymbol{A}|$ along column $j$ is

$$|\boldsymbol{A}| = a_{1j}c_{1j} + \cdots + a_{nj}c_{nj}$$

## Example (computation of determinant)

- product of pivots

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{vmatrix} \begin{vmatrix} a & 0 \\ 0 & \frac{ad-bc}{a} \end{vmatrix} \begin{vmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{vmatrix} = a \left( \frac{ad-bc}{a} \right)$$
$$= ad - bc$$

- cofactor expansion

$$\begin{vmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix}$$
$$= 2(-1)^2 \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} + (-1)(-1)^3 \begin{vmatrix} -1 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix}$$
$$= 2(2 \cdot 3 + (-1)^3(-1)(-2)) + ((-1) \cdot 3 + (-1)^3 \cdot 0)$$
$$= 2 \cdot 4 + (-3) = 5$$

## Application of Determinant

## Lemma (an identity)

*Let $\boldsymbol{A}$ be a square matrix and $\boldsymbol{C}$ be the cofactor matrix of $\boldsymbol{A}$. Then*

$$\boldsymbol{A}\boldsymbol{C}^T = |\boldsymbol{A}|\,\boldsymbol{I}$$

$$\left(\boldsymbol{A}\boldsymbol{C}^T\right)_{ij} = \sum_k a_{ik}c_{jk} = a_{i1}c_{j1} + \cdots + a_{in}c_{jn}$$

- Cofactor expansion of $|\boldsymbol{A}_{\boldsymbol{a}_{j:}\leftarrow\boldsymbol{a}_{i:}}|$ along row $j$
- $\boldsymbol{A}_{\boldsymbol{a}_{j:}\leftarrow\boldsymbol{a}_{i:}}$ has identical rows for $i \neq j$
- $\boldsymbol{A}_{\boldsymbol{a}_{j:}\leftarrow\boldsymbol{a}_{i:}} = \boldsymbol{A}$ for $i = j$

So

$$\left(\boldsymbol{A}\boldsymbol{C}^T\right)_{ij} = |\boldsymbol{A}_{\boldsymbol{a}_{j:}\leftarrow\boldsymbol{a}_{i:}}| = |\boldsymbol{A}|\,\delta_{ij}$$

That is

$$\boldsymbol{A}\boldsymbol{C}^T = |\boldsymbol{A}|\,\boldsymbol{I}$$

## Theorem (matrix inverse by cofactor matrix)

*Let $\boldsymbol{A}$ be a non-singular matrix and $\boldsymbol{C}$ be the cofactor matrix of $\boldsymbol{A}$. Then*

$$\boldsymbol{A}^{-1} = |\boldsymbol{A}|^{-1}\boldsymbol{C}^T$$

$$\begin{aligned}
& \boldsymbol{A}\boldsymbol{C}^T = |\boldsymbol{A}|\,\boldsymbol{I} \\
\Rightarrow\ & |\boldsymbol{A}|^{-1}\left(\boldsymbol{A}\boldsymbol{C}^T\right) = |\boldsymbol{A}|^{-1}|\boldsymbol{A}|\,\boldsymbol{I} = \boldsymbol{I} \\
& |\boldsymbol{A}|^{-1}\left(\boldsymbol{A}\boldsymbol{C}^T\right) = \boldsymbol{A}\left(|\boldsymbol{A}|^{-1}\boldsymbol{C}^T\right) \\
\Rightarrow\ & \boldsymbol{A}\left(|\boldsymbol{A}|^{-1}\boldsymbol{C}^T\right) = \boldsymbol{I} \\
\Rightarrow\ & \boldsymbol{A}^{-1} = |\boldsymbol{A}|^{-1}\boldsymbol{C}^T
\end{aligned}$$

## Example (inverse by cofactor matrix)

Let $\boldsymbol{A}$ be $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Since $\boldsymbol{A}$ is non-singular, we have

$$\boldsymbol{A}^{-1} = |\boldsymbol{A}|^{-1}\boldsymbol{C}^T = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

## COFACTOR MATRIX OF A NON-SINGULAR MATRIX

Let $\boldsymbol{A}$ be a non-singular matrix and $\boldsymbol{C}$ be the cofactor matrix of $\boldsymbol{A}$. Then $\boldsymbol{C}$ is a non-singular matrix.

It follows from the identity $\boldsymbol{A}\boldsymbol{C}^T = |\boldsymbol{A}|\boldsymbol{I}$ that

$$|\boldsymbol{A}|\left|\boldsymbol{C}^T\right| = |\boldsymbol{A}|^n$$

Hence

$$\left|\boldsymbol{C}^T\right| = |\boldsymbol{A}|^{n-1} \neq 0$$

and $\boldsymbol{C}$ is non-singular.

## Cofactor matrix of a singular matrix

Let $\boldsymbol{A}$ be a singular matrix and $\boldsymbol{C}$ be the cofactor matrix of $\boldsymbol{A}$. Then $\boldsymbol{C}$ is a singular matrix.

If $\boldsymbol{A} = \mathbf{0}$, then $\boldsymbol{C} = \mathbf{0}$ which is singular. If $\boldsymbol{A} \neq \mathbf{0}$, let $\boldsymbol{a}_{i:}$ be a non-zero row of $\boldsymbol{A}$. Then

$$\begin{aligned} \boldsymbol{A}\boldsymbol{C}^T = |\boldsymbol{A}|\boldsymbol{I} &\Rightarrow \boldsymbol{A}\boldsymbol{C}^T = \mathbf{0} \\ &\Rightarrow \boldsymbol{a}_{i:}\boldsymbol{C}^T = \mathbf{0} \\ &\Rightarrow \sum_j a_{ij}\boldsymbol{c}_{j:}^T = \mathbf{0} \end{aligned}$$

Since $\boldsymbol{a}_{i:} \neq \mathbf{0}$, this is a non-trivial linear combination of the rows of $\boldsymbol{C}^T$. Thus, the rows of $\boldsymbol{C}^T$ are linearly dependent and $\boldsymbol{C}$ is not of full rank. Hence $\boldsymbol{C}$ is singular.

### Theorem (Cramer's rule)

*Let* $\boldsymbol{Ax} = \boldsymbol{b}$ *be a non-singular system of linear equations. The solution is*

$$x_j = \frac{|\boldsymbol{A}_{\boldsymbol{a}_j \leftarrow \boldsymbol{b}}|}{|\boldsymbol{A}|}, \quad j = 1, \ldots, n$$

$$x_j = \frac{\begin{vmatrix} \boldsymbol{a}_1 & \ldots & \boldsymbol{a}_{j-1} & \color{red}{\boldsymbol{b}} & \boldsymbol{a}_{j+1} & \ldots & \boldsymbol{a}_n \end{vmatrix}}{\begin{vmatrix} \boldsymbol{a}_1 & \ldots & \boldsymbol{a}_{j-1} & \color{red}{\boldsymbol{a}_j} & \boldsymbol{a}_{j+1} & \ldots & \boldsymbol{a}_n \end{vmatrix}}$$

**Proof.**

Since $\boldsymbol{A}$ is non-singular, we have $\boldsymbol{A}^{-1} = |\boldsymbol{A}|^{-1}\boldsymbol{C}^T$ and the solution of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b} = |\boldsymbol{A}|^{-1}\boldsymbol{C}^T\boldsymbol{b}$. Thus

$$\begin{aligned} x_j &= |\boldsymbol{A}|^{-1}\left(\boldsymbol{C}^T\boldsymbol{b}\right)_j \\ &= |\boldsymbol{A}|^{-1}\sum_{i=1}^{n} c_{ij}b_i \\ &= |\boldsymbol{A}|^{-1} \overbrace{(b_1c_{1j} + \cdots + b_nc_{nj})}^{\text{expansion of } |\boldsymbol{A}_{\boldsymbol{a}_j \leftarrow \boldsymbol{b}}| \text{ along column } j} \\ &= \frac{|\boldsymbol{A}_{\boldsymbol{a}_j \leftarrow \boldsymbol{b}}|}{|\boldsymbol{A}|} \end{aligned}$$

$\square$

## Example (Cramer's rule)

Solve

$$\begin{array}{rcrcl} x_1 & + & 3x_2 & = & 0 \\ 2x_1 & + & 4x_2 & = & 6 \end{array}$$

$$x_1 = \frac{\begin{vmatrix} 0 & 3 \\ 6 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}} = 9, \quad x_2 = \frac{\begin{vmatrix} 1 & 0 \\ 2 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}} = -3$$

## VOLUME AND DETERMINANT

Let $\boldsymbol{A}$ be a square matrix with linearly independent row vectors. Consider box $\mathbb{B}$ formed by the row vectors of $\boldsymbol{A}$.

- For orthogonal row vectors, we have

$$(\text{volume of } \mathbb{B}) = \big|\,|\boldsymbol{A}|\,\big|$$

- For non-orthogonal row vectors, we still have

$$(\text{volume of } \mathbb{B}) = \big|\,|\boldsymbol{A}|\,\big|$$

Suppose the row vectors are orthogonal. Let $l_i$ be the length of row $i$.

$$\boldsymbol{A}\boldsymbol{A}^T = \text{diag}(l_1^2, \ldots, l_n^2) \;\Rightarrow\; \left|\boldsymbol{A}\boldsymbol{A}^T\right| = l_1^2 \,\ldots\, l_n^2$$

$$\Rightarrow\; (\text{volume of } \mathbb{B}) = l_1 \,\ldots\, l_n = \sqrt{\left|\boldsymbol{A}\boldsymbol{A}^T\right|} = \sqrt{|\boldsymbol{A}|^2} = \big|\,|\boldsymbol{A}|\,\big|$$

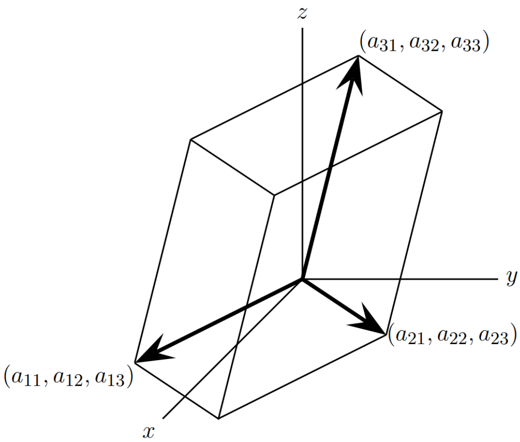


**Figure 4.1:** The box formed from the rows of $A$: volume $= |\text{determinant}|$.

Suppose the row vectors are not orthogonal.

- We find the projection of a row vector to the subspace spanned by the row vectors above it
- Then we subtract the projection from the row vector so they are orthogonal

Note

- Subtracting projection is equivalent to row operation
- The volume is invariant to subtracting projection
- The determinant is invariant to row operation

So we still have

$$(\text{volume of } \mathbb{B}) = \left| \, |\boldsymbol{A}| \, \right|$$

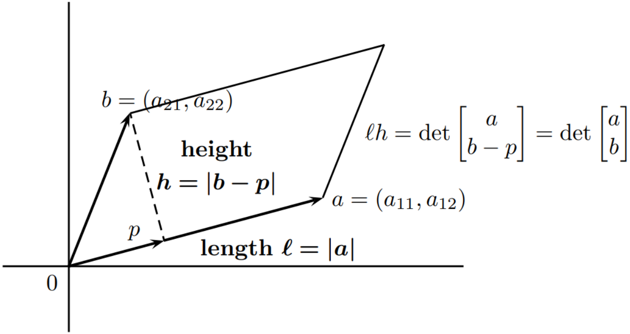


**Figure 4.2:** Volume (area) of the parallelogram $= \ell$ times $h = |\det A|$.