Positive Definite Matrix

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Linear Algebra
Notation

- $x^T Ax$: quadratic form
- $f(x)$: a multi-variate function
- $\nabla f(x)$: the gradient vector of $f(x)$
- $H(x)$: Hessian matrix of a multi-variate function
- $\sigma$: a singular value
- $\Sigma$: a singular value matrix
- $A = U\Sigma V^T$: the singular value decomposition of $A$
- $A^+$: pseudo-inverse of $A$
Quadratic Function and Quadratic Form
**Definition.** A function is **quadratic** if it is a sum of the second-order terms.

Let \( f(x_1, \ldots, x_n) \) be quadratic. Then

\[
 f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_i x_j
\]
Let $f(x_1, \ldots, x_n)$ be quadratic. Then $f(x_1, \ldots, x_n) = x^T B x$ where $x = (x_1, \ldots, x_n)^T$ and $B$ is a matrix of coefficients.

We have

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_i x_j$$

$$= x^T B x$$

where

$$b_{ij} = c_{ij}$$
**Definition.** Let $A$ be a real symmetric matrix. The **quadratic form** of $A$ is $x^T A x$.

Example. The quadratic form of

$$
A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}
$$

is

$$
x^T A x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a x_1^2 + 2 b x_1 x_2 + c x_2^2
$$
**Definition.** Let $A$ be a real symmetric matrix. Then $A$ is **positive definite** if

$$x^T A x > 0 \text{ for any } x \neq 0$$

That is, the quadratic form of $A$ is always positive except for $x = 0$, in which case $x^T A x = 0$. 
Let \( A \) be positive definite. Then the eigenvalues of \( A \) are positive.

**Proof.** Let \( \lambda_0 \) be an eigenvalue of \( A \) and \( s \) be a corresponding eigenvector. Then

\[
As = \lambda_0 s
\]

It follows that

\[
s^T As = \lambda_0 (s^T s)
\]

Hence

\[
\lambda_0 = \frac{s^T As}{s^T s} > 0
\]
Let $A$ be a real symmetric matrix. If every eigenvalue of $A$ is positive, then $A$ is positive definite.

**Proof.** By the spectral theorem, we have $A = Q\Lambda Q^T$. Consider the quadratic form

$$x^T A x = x^T Q \Lambda Q^T x = y^T \Lambda y = \sum_i \lambda_i y_i^2$$

Hence $A$ is positive definite since $x^T A x > 0$ for $x \neq 0$. 
Let $A$ be positive definite. Then every leading principal sub-matrix of $A$ has a positive determinant.

**Proof.** Let $k < n$ and consider $\mathbf{x}^T = \begin{bmatrix} \mathbf{x}_k^T & 0^T \end{bmatrix}$ with $\mathbf{x}_k \in \mathbb{R}^k$. For any $\mathbf{x}_k \neq 0$

$$\mathbf{x}^T A \mathbf{x} = \begin{bmatrix} \mathbf{x}_k^T & 0^T \end{bmatrix} \begin{bmatrix} A_k & B \\ B^T & C \end{bmatrix} \begin{bmatrix} \mathbf{x}_k^T \\ 0 \end{bmatrix} = \mathbf{x}_k^T A_k \mathbf{x}_k > 0$$

So $A_k$, the leading principle sub-matrix of $A$ of order $k \times k$, is positive definite. It follows that the eigenvalues of $A_k$ are positive, and

$$|A_k| = \prod_{i=1}^{k} \lambda_i^{(k)} > 0$$
Let $A$ be a real symmetric matrix. If every leading principal sub-matrix of $A$ has a positive determinant, then $A$ has full positive pivots.

**Proof.** By assumption $|A| > 0$, so $A$ is non-singular. Let $A = LDU$ be the LDU decomposition of $A$. Explicitly

$$
\begin{bmatrix}
A_k & B \\
B^T & C
\end{bmatrix} =
\begin{bmatrix}
L_k & 0 \\
0 & *
\end{bmatrix}
\begin{bmatrix}
D_k & 0 \\
0 & *
\end{bmatrix}
\begin{bmatrix}
U_k & * \\
0 & *
\end{bmatrix} =
\begin{bmatrix}
L_k D_k U_k & * \\
* & *
\end{bmatrix}
$$

So $A_k = L_k D_k U_k$ and $|A_k| = |D_k| = d_1 \ldots d_k$ where $d_i$ is a pivot. Thus

$$
d_1 = a_{11} > 0, \quad d_k = \frac{|A_k|}{|A_{k-1}|} > 0, \quad k = 2, \ldots, n
$$
Let $A$ be a real symmetric matrix. If $A$ has full positive pivots, then $A$ is positive definite.

**Proof.** $A$ has full pivots, so $A$ is non-singular. Let $A = LDU$ be the LDU decomposition of $A$. Since $A$ is symmetric, $A = A^T$ or $LDU = U^TDL^T$, so $U = L^T$. Thus

$$A = LDL^T = LD^{1/2}D^{1/2}L^T = R^TR$$

where $R = D^{1/2}L^T$ is non-singular. It follows that

$$x^TAx = x^TR^TRx = (Rx)^T(Rx) = \|Rx\|^2 > 0 \text{ for } x \neq 0$$

Hence $A$ is positive definite.
Equivalent Statements for PDM

Let \( A \) be a real symmetric matrix. The following statements are equivalent.

1. \( A \) is positive definite.
2. The eigenvalues of \( A \) are positive.
3. The determinants of the leading principal submatrices of \( A \) are positive.
4. The pivots of \( A \) are positive.

What we have shown in the previous slides are

\[ 1 \iff 2 \]

and

\[ 1 \implies 3 \implies 4 \implies 1 \]
The quadratic form of \( A \) is

\[ x^T A x = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 \]

\[ = 2 \left( x_1 - \frac{1}{2}x_2 \right)^2 + \frac{3}{2} \left( x_2 - \frac{2}{3}x_3 \right)^2 + \frac{4}{3}x_3^2 \]

The eigenvalues, the determinants, and the pivots are

\[
\text{spectrum}(A) = \{2, 2 \pm \sqrt{2}\}, \quad |A_1| = 2, \quad |A_2| = 3, \quad |A_3| = 4
\]
Ellipsoid

Let $A$ be positive definite. Then the equation $x^T A x = 1$ is an ellipsoid.

Explanation. By the spectral theorem, we have $A = QQ^T$. Note that $Q = \{q_1, \ldots, q_n\}$ is an orthonormal basis, and the representation of $x$ with $Q$ is $y = Q^T x$. Thus $x^T A x = 1$ can be converted to

$$x^T QQ^T x = y^T \Lambda y = \sum_i \lambda_i y_i^2 = 1$$

This is an ellipsoid with the axes of symmetry along $q_i$'s, with the intercepts of

$$y_i = \pm \left(\sqrt{\lambda_i}\right)^{-1}$$
An Ellipse

Figure 6.2: The ellipse $x^T A x = 5u^2 + 8uv + 5v^2 = 1$ and its principal axes.

$Q = \frac{1}{3} \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$

$P = \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)$
Let $A$ be a real symmetric matrix.

- $A$ is **negative definite** if $x^T A x < 0$ for any $x \neq 0$.
- $A$ is **positive semi-definite** if $x^T A x \geq 0$ for any $x$.
- $A$ is **negative semi-definite** if $x^T A x \leq 0$ for any $x$. 
Approximation and Extremal Points
Definition. Let $f(x_1, \ldots, x_n)$ be a multi-variate function. A **first-order partial derivative** of $f$ is

$$f_{x_i} = \frac{\partial f}{\partial x_i}, \; i = 1, \ldots, n$$

A **second-order partial derivative** of $f$ is

$$f_{x_ix_j} = \frac{\partial f_{x_i}}{\partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}, \; i, j = 1, \ldots, n$$

Note that

$$f_{x_ix_j} = f_{x_jx_i}$$
**Definition.** The **gradient** of $f(x_1, \ldots, x_n)$ is a vector of functions

$$\nabla f = \begin{bmatrix} f_{x_1} \\ \vdots \\ f_{x_n} \end{bmatrix}$$

The **Hessian** of $f(x_1, \ldots, x_n)$ is a matrix of functions

$$H = \begin{bmatrix} f_{x_1x_1} & \cdots & f_{x_1x_n} \\ \vdots & \ddots & \vdots \\ f_{x_nx_1} & \cdots & f_{x_nx_n} \end{bmatrix}$$
Definition. The first-order approximation to \( f(x) \) near a point \( x_0 \) is

\[
f(x) \approx f(x_0) + \nabla f(x_0)^T (x - x_0)
\]

The second-order approximation to \( f(x) \) near \( x_0 \) is

\[
f(x) \approx f(x_0) + \nabla f(x_0)^T (x - x_0) + \frac{1}{2} (x - x_0)^T H(x_0)(x - x_0)
\]
**Definition.** A point \( x_0 \) is a **stationary point** of \( f(x) \) if

\[
\nabla f(x_0) = 0
\]

Let \( x_0 \) be a stationary point of \( f(x) \). Then the second-order approximation to \( f(x) \) near \( x_0 \) is

\[
f(x) \approx f(x_0) + \frac{1}{2}(x - x_0)^T H(x_0)(x - x_0)
\]
Example

Find the second-order approximation near \((0, 0)\) to

\[
f(x, y) = 2x^2 + 4xy + y^2
\]

\[
\nabla f = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} 4x + 4y \\ 4x + 2y \end{bmatrix}, \quad H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 2 \end{bmatrix}
\]

\[
f(0) = 0, \quad \nabla f(0) = 0, \quad H(0) = \begin{bmatrix} 4 & 4 \\ 4 & 2 \end{bmatrix}
\]

\[
f(x) \approx f(0) + \frac{1}{2}(x - 0)^T H(0)(x - 0) = 2x^2 + 4xy + y^2
\]
Example

Find the second-order approximation near \((0, 0)\) to

\[ F(x, y) = 7 + 2(x + y)^2 - y \sin y - x^3 \]

\[
\nabla F = \begin{bmatrix} F_x \\ F_y \end{bmatrix} = \begin{bmatrix} 4(x + y) - 3x^2 \\ 4(x + y) - \sin y - y \cos y \end{bmatrix}
\]

\[
H = \begin{bmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{bmatrix} = \begin{bmatrix} 4 - 6x & 4 \\ 4 & 4 - 2 \cos y + y \sin y \end{bmatrix}
\]

\[ F(0) = 7, \quad \nabla F(0) = 0, \quad H(0) = \begin{bmatrix} 4 & 4 \\ 4 & 2 \end{bmatrix} \]

\[ F(x) \approx F(0) + \frac{1}{2} (x - 0)^T H(0) (x - 0) = 7 + 2x^2 + 4xy + y^2 \]
Definition.

- A point $x_0$ is a local minimum of $f(x)$ if $f(x) \geq f(x_0)$ for every $x$ in a small neighborhood of $x_0$.
- A point $x_0$ is a local maximum of $f(x)$ if $f(x) \leq f(x_0)$ for every $x$ in a small neighborhood of $x_0$. 
Let $x_0$ be a stationary point of $f(x)$.

- $x_0$ is **local minimum** if $H(x_0)$ is positive definite.
- $x_0$ is **local maximum** if $H(x_0)$ is negative definite.
- $x_0$ is a **saddle point** if it is neither local maximum nor local minimum.

For example, $(0, 0)$ is a saddle point of $F(x, y)$. 
Bowl and Saddle

Figure 6.1: A bowl and a saddle: Definite $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and indefinite $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. 
Singular Value Decomposition
Let $A$ be a real matrix. Then both matrices $(A^T A)$ and $(AA^T)$ are real symmetric and positive semi-definite.

$$x^T (A^T A) x = (Ax)^T (Ax) = \|Ax\|^2 \geq 0$$

$$x'^T (AA^T) x' = (A^T x')^T (A^T x') = \|A^T x'\|^2 \geq 0$$

**Note.** It follows that the eigenvalues of $(A^T A)$ and $(AA^T)$ are real and non-negative.
Definition. Let $A$ be a real matrix. A **singular value** of $A$ is the square root of a positive eigenvalue of $(A^T A)$. A **singular vector** of $A$ is an eigenvector of $(A^T A)$ with a positive eigenvalue.

Let $\sigma$ be a singular value of $A$. Then $\sigma > 0$. Furthermore, there exists $\nu \neq 0$ and $u \neq 0$ such that

$$(A^T A) \nu = \sigma^2 \nu$$
Definition. A right singular vector of $A$ with singular value $\sigma$ is a vector $v \neq 0$ such that

$$\left( A^T A \right) v = \sigma^2 v$$

A left singular vector of $A$ with singular value $\sigma$ is a vector $u \neq 0$ such that

$$\left( AA^T \right) u = \sigma^2 u$$
Let $A$ be a real matrix of rank $r$. Then the right singular vectors of $A$ lie in a subspace of dimension $r$.

**Proof.** Let $A$ be of order $m \times n$ with rank $r$. Note

$$
(A^T A) x = 0 \Rightarrow x^T A^T A x = 0 \Rightarrow A x = 0 \Rightarrow (A^T A) x = 0
$$

Thus

$$
\mathcal{N}(A^T A) = \mathcal{N}(A), \quad \dim \mathcal{N}(A) = n - r = \dim \mathcal{N}(A^T A)
$$

Matrix $(A^T A)$ is non-defective, so the algebraic multiplicity of eigenvalue 0 is $(n - r)$. It follows that the total algebraic (and geometric) multiplicities of the other eigenvalues of $(A^T A)$ is

$$
n - (n - r) = r$$
**Definition.** Let $A$ be a real matrix of order $m \times n$. A singular value decomposition of $A$ is

$$A = U \Sigma V^T$$

where $\Sigma$ is an $m \times n$ "diagonal" matrix with the singular values of $A$ as the leading diagonal elements, $U$ is an $m \times m$ orthogonal matrix with the eigenvectors of $\left( AA^T \right)$ as columns, and $V$ is an $n \times n$ orthogonal matrix with the eigenvectors of $\left( A^T A \right)$ as columns.
Construction of SVD

Let $r$ be the rank of $A$ and $\sigma_1 \ldots \sigma_r$ be the singular values of $A$. Let $v_1 \ldots v_r$ be orthonormal eigenvectors of $(A^T A)$ with positive eigenvalues $\sigma_i^2$ and $u_1 \ldots u_r$ be $u_i = \frac{Av_i}{\sigma_i}$. Note

$$(AA^T) u_i = \frac{AA^T Av_i}{\sigma_i} = \frac{A\sigma_i^2 v_i}{\sigma_i} = \sigma_i^2 u_i, \quad i = 1, \ldots, r$$

So $u_i$ is an eigenvector of $(AA^T)$ with the same eigenvalue $\sigma_i^2$. Let $v_{r+1} \ldots v_n$ be orthonormal eigenvectors of $(A^T A)$ with eigenvalue 0 and $u_{r+1} \ldots u_m$ be eigenvectors of $(AA^T)$ with eigenvalue 0. Construct matrices $U$ and $V$ by

$$U = \begin{bmatrix} u_1 & \ldots & u_m \end{bmatrix}, \quad V = \begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix}$$
We show $U^T AV = \Sigma$ which implies SVD $A = U \Sigma V^T$. For $j = 1 \ldots r$, we have $u_j = \frac{Av_j}{\sigma_j}$, so $Av_j = \sigma_j u_j$ and

$$
(U^T AV)_{ij} = u_i^T Av_j = u_i^T (\sigma_j u_j) = \sigma_j u_i \delta_{ij}, \ i = 1 \ldots m
$$

For $j = r + 1 \ldots n$, we have $(A^T A)v_j = 0$, so $Av_j = 0$ and

$$
(U^T AV)_{ij} = u_i^T Av_j = 0, \ i = 1 \ldots m
$$

Combining the results, we get

$$
U^T AV = \Sigma
$$

Hence

$$
A = U \Sigma V^T
$$
Matrices in an SVD

Let $A$ be a real matrix of order $m \times n$ with SVD $A = U\Sigma V^T$. Then the column vectors of $U$ (resp. $V$) is an orthonormal basis of $\mathbb{R}^m$ (resp. $\mathbb{R}^n$).

$U$ must be an eigenvector matrix of $AA^T$.

$$AA^T = (U\Sigma V^T)(V\Sigma^T U^T) = U\left(\Sigma\Sigma^T\right)U^T$$

Similarly, $V$ must be an eigenvector matrix of $A^T A$.

$$A^T A = (V\Sigma^T U^T)(U\Sigma V^T) = V\left(\Sigma^T\Sigma\right)V^T$$
Let $A$ be a real matrix with SVD $A = U\Sigma V^T$. Then the right (resp. left) singular vectors in $V$ (resp. $U$) form an orthonormal basis of $C(A^T)$ (resp. $C(A)$).

Let $A$ be of order $m \times n$ and rank $r$. \{$v_{r+1}, \ldots, v_n$\} contains eigenvectors of $(A^T A)$ with eigenvalue 0, so it is a basis of $N(A^T A) = N(A)$. Hence \{$v_1, \ldots, v_r$\} is a basis of the orthogonal complement of $N(A)$, i.e. $C(A^T)$.

**Column space.** From $AV = U\Sigma$, we have

$$Av_i = \sigma_i u_i, \quad i = 1, \ldots, r$$

Thus $u_1, \ldots, u_r$ are vectors in $C(A)$. Furthermore, since they are linearly independent, they form a basis of $C(A)$. 

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Positive Definite Matrix
Example

Find SVD of

\[ A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \]

The eigenvalues of

\[ A^T A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \]

are \( \lambda_1 = 3, \lambda_2 = 1, \lambda_3 = 0 \). Hence the singular values of \( A \) are

\[ \sigma_1 = \sqrt{3}, \sigma_2 = 1 \]
Orthonormal eigenvectors of \((A^TA)\) are

\[
\begin{align*}
v_1 &= \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \\
v_2 &= \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \\
v_3 &= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\end{align*}
\]

The corresponding left singular vectors of \(A\) are

\[
\begin{align*}
u_1 &= \frac{A v_1}{\sigma_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \\
u_2 &= \frac{A v_2}{\sigma_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}
\end{align*}
\]

So

\[
A = U\Sigma V^T = \begin{bmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\sqrt{3} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}
\end{bmatrix}
\]
Let $A$ be a real matrix of rank $r$. Then $A$ can be expressed as the sum of $r$ real matrices of rank 1 based on singular values and singular vectors.

By SVD

$$A = U \Sigma V^T = \sigma_1 u_1 v_1^T + \cdots + \sigma_r u_r v_r^T = A_1 + \cdots + A_r$$

**Image approximation.** For an image of size $1000 \times 1000$, a compression rate of 90% is achieved if 50 terms are used.

*Data Compression with SVD*
Let $A$ be a real matrix with SVD $A = U\Sigma V^T$. Then the minimum-length least-squares solution to $Ax = b$ is $x^+ = V\Sigma^+ U^T b$.

**Pseudo-inverse.** The minimum-length least-squares solution can be written as $x^+ = A^+ b$ where $A^+ = V\Sigma^+ U^T$. $A^+$ is called the **pseudo-inverse** of $A$. 

Figure 3.4: The true action $Ax = A(x_{row} + x_{null})$ of any $m$ by $n$ matrix.

Figure 6.3: The pseudoinverse $A^+$ inverts $A$ where it can on the column space.
Minimum Principles
Non-singular System of Linear Equations

Let $A$ be positive definite. Then $x_0$ achieves the minimum of

$$P(x) = \frac{1}{2}x^T Ax - x^T b$$

if and only if $Ax_0 = b$.

Note

$$\nabla (x^T Ax) = 2Ax, \quad \nabla (x^T b) = b$$

so

$$\nabla P = Ax - b$$
Figure 6.4: The graph of a positive quadratic $P(x)$ is a parabolic bowl.
Figure 6.5: Minimizing $\frac{1}{2} \|x\|^2$ for all $x$ on the constraint line $2x_1 - x_2 = 5$. 

$P = \frac{1}{2}(x_1^2 + x_2^2)$

$P_{C/\text{min}} = \frac{5}{2}$
Over-determined System

Let $\mathcal{L} : Ax = b$ be an over-determined system of linear equations. The sum of squared errors as a function of $x$ is

$$E^2(x) = \| Ax - b \|^2$$

$$= (Ax - b)^T (Ax - b)$$

$$= x^T A^T Ax - 2x^T A^T b + b^T b$$

An $x_0$ that achieves the minimum of $E^2(x)$ satisfies

$$\nabla E^2(x) \bigg|_{x=x_0} = 0$$

That is

$$A^T Ax_0 = A^T b$$
Let $A$ be a real symmetric matrix of order $n$ with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$. Then

$$R(x) = \frac{x^T A x}{x^T x} \geq \lambda_1$$

Furthermore, $\lambda_1$ is achieved by $s_1 \in \mathbb{E}_{\lambda_1}$.

- $R(x)$ is called the Rayleigh quotient.
- We also have $R(x) \leq \lambda_n$, and $\lambda_n$ is achieved by $s_n \in \mathbb{E}_{\lambda_n}$. 
The Rayleigh quotient is $a_{ii}$ when $x = e_i$. So

$$\lambda_1 \leq a_{ii} \leq \lambda_n$$

which implies

$$\frac{1}{\sqrt{\lambda_n}} \leq \frac{1}{\sqrt{a_{ii}}} \leq \frac{1}{\sqrt{\lambda_1}}$$

**Figure 6.6:** The farthest $x = x_1/\sqrt{\lambda_1}$ and the closest $x = x_n/\sqrt{\lambda_n}$ both give $x^T Ax = x^T \lambda x = 1$. These are the major axes of the ellipse.
Intermediate Eigenvalues

The minimum of $R(x)$ subject to $x^T s_1 = 0$ is $\lambda_2$. For any $v$, the minimum of $R(x)$ subject to $x^T v = 0$ cannot be above $\lambda_2$. That is

$$\lambda_2 \geq \min_{x^T v = 0} R(x)$$

This gives us the maximin principle for $\lambda_2$ as follows

$$\lambda_2 = \max_v \left[ \min_{x^T v = 0} R(x) \right]$$

More generally, let $S_j$ be a subspace of dimension $j$, then

$$\lambda_{j+1} = \max_{S_j} \left[ \min_{x \perp S_j} R(x) \right] , \quad \lambda_{n-j} = \min_{S_j} \left[ \max_{x \perp S_j} R(x) \right]$$