Digital Signal Processing - II

Discrete Fourier Transform

- The **discrete Fourier transform** (DFT) of a sequence of finite duration N is defined by

$$X[k] = \sum_{n=0}^{N-1} x[n]W^{nk}, \quad 0 \le k < N, \quad W = e^{-j(2\pi/N)} \tag{1}$$

- The inverse transform (IDFT) is given by

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W^{-kn}, \quad 0 \le n < N. \tag{2}$$

- The above equality can be generalized for the help of the following equality

$$\sum_{k=0}^{N-1} W^{ck}W^{-nk} = N\delta_{cn}. \tag{3}$$

- For $x[n]$ of finite duration N, the $X[k]$'s are exactly the N equally spaced samples at points $\omega_k = k\frac{2}{N}$ of $X(e^{j\omega})$

Signal Flow Graph

- A discrete-time LTI system is also called a **digital filter**.
- From $y[n] = x[n] * h[n]$, the input-output relation of a digital filter can also be expressed in the z-domain as

$$Y(z) = X(z)H(z), \tag{4}$$

or in the ω-domain as

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}), \tag{5}$$

- $H(z)$ is called the *system function* or *transfer function*.
- $H(e^{j\omega})$ is called the *frequency response*.

Signal Flow Graph

- Proof of (4)

$$\begin{aligned} Y(z) &= \sum_{\omega} y[n]z^{-n} = \sum_{\omega}\sum_{m} x[m]h[n-m]z^{-n} \\ &= \sum_{m} x[m]z^{-m}\sum_{\omega} h[n-m]z^{-(n-m)}z^{-m} \\ &= \sum_{\omega} x[m]z^{-m}\sum_{\omega} h[n]z^{-n} = X(z)H(z). \end{aligned} \tag{6}$$

Causal Systems and Stable Systems

- A system is **causal** if its impulse response is zero for negative n, i.e.,

$$h[n] = 0, \quad \forall n < 0. \tag{7}$$

  - Note that the output signal does not exist before the input signal.
- A system is **stable** if a bounded input signal produces a bounded output signal (BIBO).
  - A necessary and sufficient condition for stability is

$$\sum_{n} |h[n]| < \infty, \tag{8}$$

  which also sufficient for the existence of $H(e^{j\omega})$.

Minimum-Phase

- We have the convolution theorem

$$y[n] = x[n] * h[n] \;\Rightarrow\; Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) \tag{9}$$

- The dual is the **modulation theorem**

$$y[n] = x[n]h[n] \;\Rightarrow\; Y(e^{j\omega}) = X(e^{j\omega}) * H(e^{j\omega}) \tag{10}$$

Poles and Zeros

- A **pole** of a function is where the value of the function is singular
- A **zero** of a function is where the value of the function is 0
- When a system function is expressed as the ratio of two polynomials
  - the roots of the numerator are zeros
  - the roots of the denominator are poles

Linear Difference Equation

- A causal LTI system is sometimes characterized by a **linear difference equation**

$$y[n] - \sum_{k=1}^{N} a_k y[n-k] = \sum_{r=0}^{M} b_r x[n-r]. \tag{11}$$

Transformation Analysis

- Noting the z-transform of $y[n-k]$ is $z^{-k}Y(z)$, from (11) we have

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{r=0}^{M} b_r z^{-r}}{1 - \sum_{k=1}^{N} a_k z^{-k}} \tag{12}$$

- In terms of the zeros and the poles, (12) can be re-written as

$$H(z) = \frac{A\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}, \tag{13}$$

  where c_k's are the zeros and d_k's are the poles.
- Note that for a stable and causal system, all d_k's must be inside the unit circle.

Finite Impulse Response

- If the a_k's in (11) are zero, we have

$$y[n] = \sum_{r=0}^{M} b_r x[n-r]. \tag{14}$$

- This corresponds to an FIR filter with

$$h[n] = \begin{cases} b_n, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases} \tag{15}$$

Linear Phase and Sine Delay

- Consider the ideal delay system

$$y[n] = x[n - n_d]. \tag{16}$$

- The impulse response is

$$h_d[n] = \delta[n - n_d]. \tag{17}$$

- The frequency response is the Fourier transform

$$H_d(e^{j\omega}) = e^{-j\omega n_d}. \tag{18}$$

  One can see a delay introduce a linear phase change

$$\angle H_d(e^{j\omega}) = -\omega n_d. \tag{19}$$

Linear Phase FIR Filter

- An FIR filter can be made linear-phase as follows.
- Suppose $h[n] = h[M - n]$, and M is even. Denoting $\frac{M}{2}$ as α, we have

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} = \left(\sum_{n=0}^{\alpha-1} + \sum_{n=\alpha+1}^{M}\right) h[n]e^{-j\omega n} + h[\alpha]e^{-j\omega\alpha} \\ &= \sum_{k=1}^{\alpha}\left(h[\alpha - k]e^{-j\omega(\alpha-k)} + h[\alpha + k]e^{-j\omega(\alpha+k)}\right) + h[\alpha]e^{-j\omega\alpha} \\ &= \sum_{k=1}^{\alpha} h[\alpha - k]\left(e^{-j\omega(\alpha-k)} + e^{-j\omega(\alpha+k)}\right) + h[\alpha]e^{-j\omega\alpha} \\ &= \sum_{k=1}^{\alpha} h[\alpha - k]\left(e^{j\omega k} + e^{-j\omega k}\right)e^{-j\omega\alpha} + h[\alpha]e^{-j\omega\alpha} \\ &= A(\omega)e^{-j\omega\alpha} \end{aligned} \tag{20}$$

Infinite Impulse Response

- (11) can be re-written as a recurrence formula

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{r=0}^{M} b_r x[n-r]. \tag{21}$$

- Suppose $M \le N$. (12) can be expanded by partial fraction expansion

$$H(z) = A_0 + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}. \tag{22}$$

- For a causal system, it is easily shown that

$$h(n) = A_0\delta[n] + \sum_{k=1}^{N} A_k (d_k)^n u[n]. \tag{23}$$

  $h[n]$ has infinite duration. Such a filter is called infinite-impulse response (IIR).

Implementation of IIR Filters

- Direct form I and direct from II, Fig. 2.9
- Cascade form and parallel form, Fig 2.10
- Design issues are the storage, round-off noise, stability, etc.

Sampling

- A discrete-time signal often arises from **periodic sampling** of a continuous-time signal. That is,

$$x[n] = x_a(nT). \tag{24}$$

- Note that there is an important parameter, T, which is called the **sampling period**. With different sampling periods, the sampled sequences actually look quite different.
- The big questions are
  - Under what condition can we recover a continuous function $x_a(t)$ from its samples $x[n]$?
- Intuitively, the smaller T, the more information we keep about $x_a(t)$ in $x[n]$.
- Is there a small enough T such that we can reconstruct $x_a(t)$ from $x[n]$?
- Counter-intuitively, the answer is **yes**.
- If the sampling rate is **sufficiently high** relative to the **signal bandwidth**, it is theoretically possible to reconstruct a $x_a(t)$ from $x_a(nT)$.

The Sampling Theorem

- Let $x_a(t)$ be a bandlimited signal with

$$X_a(j\Omega) = 0, \quad |\Omega| \ge \Omega_N. \tag{25}$$

  Then $x_a(t)$ is uniquely determined by its samples $x[n] = x_a(nT)$, $n = 0, \pm 1, \pm 2, \ldots$, if

$$\Omega_s \ge 2\Omega_N. \tag{26}$$

- $F_N = \Omega_N/2\pi$ is called the **Nyquist frequency**. $2F_N$ is called the **Nyquist rate**, which is the minimum sampling frequency required for perfect reconstruction.
- It can be shown that

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k} X_a\left(j\left(\frac{\omega}{T} + 2\pi k\right)\right) \tag{27}$$

- Suppose the sampling frequency satisfies $F_s = \frac{1}{T} > 2F_N$ so $\Omega_s T < \pi$. Combined with (25), one can see that

$$X_a\left(j\left(\frac{\omega}{T} + 2\pi k\right)\right), \quad k = 0, \pm 1, \pm 2, \ldots \tag{28}$$

  are non-overlapping. So the period of $X(e^{j\omega})$ around $\omega = 0$ reproduces the original spectrum $X_a(j\Omega)|_{\Omega<\frac{\pi}{T}}$.
- In fact, we can reconstruct $x_a(t)$ by

$$\begin{aligned} x_a(t) &= \sum_{n} x[n]\frac{\sin(\pi(t - nT)/T)}{\pi(t - nT)/T} \\ \Rightarrow\; x_a(t) &= \sum_{n} x_a(nT)\frac{\sin(\pi(t - nT)/T)}{\pi(t - nT)/T}. \end{aligned} \tag{29}$$

Sampling Rates

- **Wideband Speech**: 15000 – 20000 Hz; 16k Hz is common
- **Telephone Bandwidth Speech**: 300 – 3200 Hz; 8k Hz is common
- **Wideband Audio**: 42000– Hz, due to auditory threshold in frequency; 44100 is common

Changing Sampling Rates

- Suppose a continuous-time signal $x_a(t)$ is represented by a discrete-time sequence

$$x[n] = x_a(nT). \tag{30}$$

- It is often necessary to change the discrete-time representation to

$$x'[n] = x_a(nT'). \tag{31}$$

  for some $T' \neq T$. This is called **changing sampling rate**.

Decimation

- When we are taking less samples, it is called decimation

$$T' = MT \tag{32}$$

- A downsampler has output-input relation

$$x_d[n] = x[nM] \tag{33}$$

- Just like $X(e^{j\omega T})$ is the sum of copies of $X_a(j\Omega)$, $X_d(e^{j\omega})$ is the sum of copies of $X(e^{j\omega})$, with shift $2\pi/M$. So a low-pass filter (in normalized frequency) can make sure that the original low-frequency part is preserved.
- Fig. 2.15

Interpolation

- When we are taking more samples, it is called interpolation.

$$T' = T/L \tag{34}$$

- An upsampler has output-input relation

$$x_e[n] = \sum_{k} x[k]\delta[n - kL] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L \ldots \\ 0, & \text{otherwise} \end{cases} \tag{35}$$

- It can be shown that

$$X_e(e^{j\omega}) = X(e^{j\omega L}) \tag{36}$$

  That is, the spectrum $X(e^{j\omega})$ in $(-\pi, \pi)$ is squeezed in $(-\pi/L, \pi/L)$
- To keep the original spectrum, a low-pass filter (in normalized frequency) is required.
- Fig. 2.18

Non-integer Sampling Rate Changes

- For non-integer sampling rate changes, we can cascade an interpolator and a decimator.
- Fig. 2.19