Digital Signal Processing - I

Introduction

- A brief review of the concepts in **digital signal processing** that are relevant to the speech/audio signals
- Condensing an entire subject into a chapter
- You may need to read DSP books to supplement the details

Discrete-Time Signals

- A continuous-time signal $x_a(t)$ is a function of time
- A discrete-time signal $x[n]$ is a sequence of numbers
- $x[n]$ can be obtained by **sampling** a continuous-time signal $x_a(t)$ at **periodic** time stamps,

$$x[n] = x_a(nT) \tag{1}$$

- There are two key parameters
  - the sampling period T
  - the number of quantization levels 2^B
- A digital representation is lossless as long as

$$\frac{1}{T} \triangleq F_S \geq 2F_N, \tag{2}$$

where F_N is the highest frequency in $x_a(t)$.
- See Fig. 2.1 for an example

Common Signals

- The **sinusoidal signal**

$$x[n] = A\cos(\omega n + \phi) \tag{3}$$

  - A is called the **amplitude**
  - ω is called the **normalized radian frequency**
  - $\omega/2\pi$ is called the **normalized cyclic frequency**
  - $(\omega n + \phi)$ is called the **phase**
  - ϕ is called the **phase shift**
  - Note that $x[n]$ may be aperiodic in n
- The **impulse signal**

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

- The **step signal**

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

- The **rectangular signal**

$$\text{rect}_N[n] = \begin{cases} 1, & 0 \leq n < N \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

- The **real exponential**

$$x[n] = a^n u[n] \tag{7}$$

- The **complex exponential**

$$x[n] = a^n u[n] = r^n e^{j\omega n} u[n] = r^n(\cos\omega n + j\,\sin\omega n)u[n] \tag{8}$$

Linear and Time-Invariant System

- A discrete-time system, given an input discrete-time signal $x[n]$, outputs a discrete-time signal $y[n]$

$$y[n] = T\{x[n]\}. \tag{9}$$

- A system is said to be **linear** if

$$\begin{aligned} &T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\} \\ &T\{\alpha x[n]\} = \alpha T\{x[n]\} \end{aligned} \tag{10}$$

- A system is said to be **time-invariant** if

$$T\{x[n]\} = y[n] \Rightarrow T\{x[n - n_0]\} = y[n - n_0]. \tag{11}$$

- A linear time-invariant (LTI) system is linear and time-invariant.

Impulse Response

- The **impulse response** of a discrete-time system is the output signal when the input signal is the impulse signal

$$h[n] = T\{\delta[n]\}. \tag{12}$$

Convolution

- The **convolution** of two discrete-time signals is

$$x[n] * h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]. \tag{13}$$

- It can be shown that the convolution is **commutative, associative** and **distributive**.

$$\begin{aligned} &x[n] * h[n] = h[n] * x[n] \\ &x[n] * (h[n] * g[n]) = (x[n] * h[n]) * g[n] \\ &x[n] * (h[n] + g[n]) = (x[n] * h[n]) + (x[n] * g[n]). \end{aligned} \tag{14}$$

Theorem

- (**Theorem**) *The output of an LTI system is the convolution of the input signal and the impulse response.*
- (**Proof**) $x[n]$ can be written as a *sum of impulse signals*

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\delta[n-m]. \tag{15}$$

It follows from the linearity and time-invariance that

$$\begin{aligned} y[n] &= T\{x[n]\} = T\left\{\sum_{m} x[m]\delta[n-m]\right\} \\ &= \sum_{m} x[m]T\{\delta[n-m]\} = \sum_{m} x[m]h[n-m]. \end{aligned} \tag{16}$$

Eigenfunction of LTI System

- If we input a **complex exponential** $x[n] = a^{z,n}$ to an LTI system, the output is still a complex exponential, but with a different amplitude

$$y[n] = \sum_{m} h[m]a^{z,n-m} = a^{z,n}H(a^{z,n}) = x[n]H(a^{z,n}) \tag{17}$$

- $a^{z,n}$ is called an **eigenfunction**, with **eigenvalue** $H(a^{z,n})$.

Continuous-Time Fourier Transform

- The **continuous-time Fourier transform (CTFT)** is defined by

$$X_a(\Omega) = \int_{-\infty}^{\infty} x_a(t)e^{-j\Omega t}dt. \tag{18}$$

- The corresponding **inverse transform** is defined by

$$x_a(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(\Omega)e^{j\Omega t}d\Omega. \tag{19}$$

- Note we use Ω to denote radian frequency. It is related to cyclic frequency (cycles per second) F by

$$\Omega = 2\pi F \tag{20}$$

z-Transform

- The **z-transform** of a discrete-time signal $x[n]$ is

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}. \tag{21}$$

- The **inverse z-transform** is

$$x[n] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz \tag{22}$$

- (22) can be proved by noting that

$$\oint_C z^k dz = \begin{cases} 2\pi j & k = -1, \\ 0 & k \neq -1. \end{cases} \tag{23}$$

Region of Convergence

- The right-side sum,

$$\sum_{n} x[n]z^{-n}. \tag{24}$$

  in (21) may diverge.
- $X(z)$ is only defined at a point z where the sum converges.
- The **region of convergence** (ROC) for a signal $x[n]$ is the region where the above sum converges.
- In general, an ROC is of the form

$$R_1 < |z| < R_2. \tag{25}$$

Examples of z-Transform

- **Delayed Impulse**

$$x[n] = \delta[n - n_0] \overset{\mathcal{Z}}{\Rightarrow} X(z) = z^{-n_0} \tag{26}$$

- **Rectangular signal** (Box Pulse)

$$h[n] = u[n] - u[n - N] \overset{\mathcal{Z}}{\Rightarrow} H(z) = \frac{1 - z^{-N}}{1 - z^{-1}}, \quad z \neq 0 \tag{27}$$

- **Right-sided exponential signal**

$$h_1[n] = a^n u[n] \overset{\mathcal{Z}}{\Rightarrow} H_2(z) = \frac{1}{1 - az^{-1}}, \quad |z| > |a| \tag{28}$$

- **Left-sided exponential signal**

$$h_1[n] = -a^n u[-n-1] \overset{\mathcal{Z}}{\Rightarrow} H_4(z) = \frac{1}{1 - az^{-1}}, \quad |z| < |a| \tag{29}$$

Discrete-time Fourier Transform

- The **Discrete-time Fourier transform (DTFT)** is

$$X(e^{j\omega}) = \sum_{n} x[n]e^{-j\omega n}. \tag{30}$$

- The inverse transform is

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega. \tag{31}$$

$x[n]$ can be seen as being **synthesized** by exponential signals $e^{j\omega}$ of various ω.
- From (17), the eigenvalue $H(e^{j\omega})$ for the eigenfunction $e^{j\omega}$ of an LTI system with impulse response $h[n]$ is

$$H(e^{j\omega}) = \sum_{n} h[n]e^{j\omega n}. \tag{32}$$

- The DTFT is a special case of the z-transform (21), i.e., on a point of the **unit circle**

$$z = e^{j\omega}, \quad |z| = 1 \tag{33}$$

- For the DTFT of a signal $x[n]$ to converge, we need

$$\sum_{n} |x[n]| < \infty \tag{34}$$

Periodic Discrete Signals

- Let

$$x[n] = \cos\omega_0 n, \quad -\infty < n < \infty \tag{35}$$

- (34) is not satisfied, and DTFT of $x[n]$ does not converge in the regular sense.
- By introducing the Dirac delta function, we can write

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \left[\pi\delta(\omega - \omega_0 + 2\pi k) + \pi\delta(\omega + \omega_0 + 2\pi k)\right], \tag{36}$$

which can be verified by the inverse transform (31).

Periodicity and Sampling of Frequency

- From (30), one can see that $X(e^{j\cdot})$ is a **periodic** function of ω, with period 2π, i.e.,

$$X(e^{j\omega}) = X(e^{j(\omega+2\pi)}) \tag{37}$$

- ω is called the **normalized radian frequency**
- Alternatively, one may find it intuitive to use the **normalized cyclic frequency** f, which is

$$f = \frac{\omega}{2\pi}. \tag{38}$$

- Consider $x_a(t)$ as a continuous-time sinusoidal signal with **unnormalized cyclic frequency** F

$$x_a(t) = \cos(2\pi Ft), \quad \Omega = 2\pi F \tag{39}$$

- Let $x[n]$ be sampled from $x_a(t)$ with period T,

$$x[n] = x_a(nT) = \cos(2\pi FnT) \tag{40}$$

- $x[n]$ is a sinusoidal signal with normalized radian frequency

$$x[n] = \cos\omega n \Rightarrow \omega = 2\pi FT, \quad f = FT. \tag{41}$$

- To sum up, these normalized and unnormalized frequencies are related by

$$f = FT \rightarrow F/F_S, \quad \omega = \Omega T = \Omega/F_S \tag{42}$$