Solution of the LPC Equations

The unknowns in the LPC equations are the linear prediction coefficients

$$\alpha_k, \quad k = 1, \ldots, p \tag{1}$$

which must satisfy a system of linear equations due to the minimization of the squared prediction errors.

Cholesky Decomposition for the Covariance Method
We have the following system of linear equations to solve

$$\sum_{k=1}^{p} \hat{c}_k \hat{\phi}_0[i,k] = \hat{\phi}_0[i,0], \quad i = 1, \ldots, p \tag{2}$$

In what follows, the short-time index $\hat{n}$ will be dropped for notational simplicity. (2) can be written in a matrix form

$$\Phi \alpha = \psi \tag{3}$$

Note that Φ is symmetric in (3), so it can be written as

$$\Phi = V D V^T, \tag{4}$$

where V is a *unit-lower triangular* matrix and D is diagonal. It follows that

$$\hat{\phi}_0 = \sum_{k=1}^{p} V_k d_k V_k, \quad \hat{\phi}_k = \sum_{k=1}^{p} V_k d_k V_k. \tag{5}$$

The entries in V and D can be computed column by column by

$$d_1 = \phi_{11}, \quad V_1 = \frac{\phi_1}{d_1} \tag{6}$$

for the first column, and subsequently

$$d_i = \hat{\phi}_i - \sum_{k=1}^{i-1} V_k^2 d_k, \quad V_i = \frac{1}{d_i}\left(\hat{\phi}_i - \sum_{k=1}^{i-1} V_k d_k V_k\right) \tag{7}$$

With (4), (3) can be re-written as

$$V D V^T \alpha = \psi, \tag{8}$$

which can be solved by

$$V Y = \psi \quad \text{and} \quad V^T \alpha = D^{-1} Y. \tag{9}$$

Since V and V^T are triangular, the solution of (10) can be computed very fastly, with

$$\begin{aligned} Y_1 &= \psi_1, \quad Y_i = \psi_i - \sum_{j=1}^{i-1} V_{ij} Y_j \\ \alpha_p &= Y_p / d_p, \quad \alpha_i = Y_i / d_i - \sum_{j=i+1}^{p} V_{ij} \alpha_j \end{aligned} \tag{10}$$

by forward and backward substitution, respectively. For the squared error, we have

$$\begin{aligned} \hat{\mathcal{E}}_0 &= \hat{\phi}_0[0,0] - \sum_{k=1}^{p} \hat{c}_k \hat{\phi}_0[0,k] = \hat{\phi}_0[0,0] - \alpha^T \psi \\ &= \hat{\phi}_0[0,0] - Y^T D^{-1} V^{-1} \psi = \hat{\phi}_0[0,0] - Y^T D^{-1} Y \\ &= \hat{\phi}_0[0,0] - \sum_{k=1}^{p} Y_k^2 / d_k \end{aligned} \tag{11}$$

Levinson-Durbin Algorithm for the Autocorrelation Method
For the autocorrelation method, we have the following system of linear equations

$$\sum_{k=1}^{p} \hat{c}_k R[|i-k|] = R[i], \quad i = 1, \ldots, p \tag{12}$$

(12) can be written in a matrix form

$$R \alpha = r \tag{13}$$

where R is a positive definite symmetric Toeplitz matrix with

$$R_{ij} = R[|i-j|], \tag{14}$$

and

$$r_i = R[i]. \tag{15}$$

For the squared error, we have

$$\hat{\mathcal{E}}^{(p)} = R[0] - \sum_{k=1}^{p} \alpha_k R[k], \tag{16}$$

where the superscript $^{(p)}$ means it is obtained from a p^{th}-order optimal predictor. (12) and (16) can be combined in a matrix equation with $p+1$ unknowns and $p+1$ equations

$$\begin{bmatrix} R[0] & R[1] & \ldots & R[p-1] & R[p] \\ R[1] & R[0] & \ldots & R[p-2] & R[p-1] \\ , & , & , & , & , \\ , & , & , & , & , \\ R[p-1] & R[p-2] & \ldots & R[0] & R[1] \\ R[p] & R[p-1] & \ldots & R[1] & R[0] \end{bmatrix} \begin{bmatrix} 1 \\ -\alpha_1^{(p)} \\ -\alpha_2^{(p)} \\ , \\ -\alpha_p^{(p)} \end{bmatrix} = \begin{bmatrix} \hat{\mathcal{E}}^{(p)} \\ 0 \\ 0 \\ , \\ 0 \end{bmatrix} \tag{17}$$

Seeing (17) as the special case of $i = p$, for any order i, we can define a system of linear equations

$$R^{(i)} \gamma_i^{(i)} = \mathcal{E}^{(i)} \tag{18}$$

The Levinson-Durbin algorithm can recursively compute $\alpha^{(i)}$ given $\alpha^{(i-1)}$.

Levinson-Durbin Recursion
Let $\alpha^{(i-1)}$ be the solution for $R^{(i-1)} \alpha^{(i-1)} = \mathcal{E}^{(i-1)}$. By appending a 0 to $\alpha^{(i-1)}$ and multiply from left by $R^{(i)}$, we have

$$\begin{bmatrix} R[0] & R[1] & \ldots & R[i-1] & R[i] \\ R[1] & R[0] & \ldots & R[i-2] & R[i-1] \\ , & , & , & , & , \\ , & , & , & , & , \\ R[i-1] & R[i-2] & \ldots & R[0] & R[1] \\ R[i] & R[i-1] & \ldots & R[1] & R[0] \end{bmatrix} \begin{bmatrix} 1 \\ -\alpha_1^{(i-1)} \\ -\alpha_2^{(i-1)} \\ , \\ -\alpha_{i-1}^{(i-1)} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathcal{E}^{(i-1)} \\ 0 \\ 0 \\ , \\ 0 \\ \gamma^{(i-1)} \end{bmatrix} \tag{19}$$

where

$$\gamma^{(i-1)} = R[i] - \sum_{j=1}^{i-1} \alpha_j^{(i-1)} R[i-j] \tag{20}$$

(19) can also be written as

$$R^{(i)} \begin{bmatrix} 1 \\ -\alpha_1^{(i-1)} \\ -\alpha_2^{(i-1)} \\ , \\ -\alpha_{i-1}^{(i-1)} \\ 0 \end{bmatrix} - k_i \begin{bmatrix} 0 \\ -\alpha_{i-1}^{(i-1)} \\ -\alpha_{i-2}^{(i-1)} \\ , \\ -\alpha_1^{(i-1)} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathcal{E}^{(i-1)} \\ 0 \\ 0 \\ , \\ 0 \\ \gamma^{(i-1)} - k_i \mathcal{E}^{(i-1)} \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 \\ -\alpha_1^{(i-1)} \\ -\alpha_2^{(i-1)} \\ , \\ -\alpha_{i-1}^{(i-1)} \\ 0 \end{bmatrix} - k_i \begin{bmatrix} 0 \\ -\alpha_{i-1}^{(i-1)} \\ -\alpha_{i-2}^{(i-1)} \\ , \\ -\alpha_1^{(i-1)} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathcal{E}^{(i-1)} - k_i \gamma^{(i-1)} \\ 0 \\ 0 \\ , \\ 0 \\ \gamma^{(i-1)} - k_i \mathcal{E}^{(i-1)} \end{bmatrix} \tag{21}$$

To make (22) become (18), we must choose k_i such that

$$\gamma^{(i-1)} - k_i \mathcal{E}^{(i-1)} = 0 \tag{22}$$

k_i is called the partial correlation coefficients (PARCOR), and given by

$$k_i = \frac{\gamma^{(i-1)}}{\mathcal{E}^{(i-1)}} = \frac{R[i] - \sum_{j=1}^{i-1} \alpha_j^{(i-1)} R[i-j]}{\mathcal{E}^{(i-1)}} \tag{23}$$

It follows that

$$\mathcal{E}^{(i)} = \mathcal{E}^{(i-1)} - k_i \gamma^{(i-1)} = \mathcal{E}^{(i-1)}(1 - k_i^2) \tag{24}$$

and

$$\alpha_j^{(i)} = \begin{cases} \alpha_j^{(i-1)} - k_i \alpha_{i-j}^{(i-1)} & j < i \\ k_i & j = i \end{cases} \tag{25}$$

Starting with $\mathcal{E}^{(0)} = R[0]$, k_i and $\alpha_j^{(i)}$ can be recursively obtained, as shown in Figure 9.18.
From (25) we have

$$\mathcal{E}^{(i)} = \mathcal{E}^{(i-1)} - k_i \gamma^{(i-1)} = \mathcal{E}^{(i-1)}(1 - k_i^2) = \mathcal{E}^{(0)} \prod_{m=1}^{i} (1 - k_m^2) \tag{26}$$

Since $\mathcal{E}^{(i)} \geq 0$, we conclude that the PARCOR coefficients satisfy

$$-1 \leq k_m \leq 1, \quad \forall\, m \tag{27}$$

Application of LPC

In this section we look at the LPC at work. First, we define the normalized prediction error for the auto-correlation method as

$$V_h = \frac{\sum_{m=0}^{L-|h|-1} e_L^2[m]}{\sum_{m=0}^{L-1} s_L^2[m]} \tag{28}$$

and for the covariance method as

$$V_h = \frac{\sum_{m=0}^{L-1} e_L^2[m]}{\sum_{m=0}^{L-1} s_L^2[m]} \tag{29}$$

where $e_L^2[m]$ is the prediction error. There are several formula for V_h in Table 9.2. One note that the autocorrelation method is the most flexible.

Spectrum of the Error Signal
Figure 9.25 - 9.28 shows that the spectrum is relatively flat, regardless of male/female, auto-correlation/covariance method.

Prediction Error as a Function of Order p
Figure 9.29 is the case of $L = 60$, a short window, while Figure 9.30 is the case of $L = 120$, a window longer than the pitch period.83 samples for a synthetic vowel. One can see that the prediction error decreases with p.
Figure 9.31 shows the cases of varying frame sizes for a synthetic vowel.

Figure 9.32 shows the cases of varying p for a natural vowel, where $L < N_p$.
Figure 9.33 shows the cases of varying p for a natural vowel, where $L > N_p$.
Figure 9.35 shows the cases of varying frame sizes for a natural speech voiced section.

The normalized error also depends on the frame position and window function, as shown in Figure 9.35.

Properties of the LPC Polynomial

The LPC polynomial (a.k.a. prediction error polynomial) is

$$A(z) = 1 - \sum_{k=1}^{p} \alpha_k z^{-k}. \tag{30}$$

The following properties are possessed by $A(z)$

1. All the zeros of $A(z)$ is inside the unit circle.
2. The formants are related to the roots of $A(z)$ closest to the unit circle. There are two ways to show the first property. See (9.138), and (9.141). For the second property, just remeber that the magnitude response is a sum of terms, each is inversely proportional to the distance to the unit circle. Figure 9.36 and 9.37 give an example.

Alternative Representations

Roots of $A(z)$
$z_1, \ldots, z_p$ in

$$A(z) = 1 - \sum_{k=1}^{p} \alpha_k z^{-k} = \prod_{k=1}^{p} (1 - z_k z^{-1}) \tag{31}$$

Impulse Response $\hat{H}(z)$
$\hat{h}[n]$ in

$$\hat{H}(z) = \frac{G}{A(z)} = \frac{G}{1 - \sum_{k=1}^{p} \alpha_k z^{-k}} \quad \Rightarrow \quad \hat{h}[n] = \sum_{k=1}^{p} A_k z_k^n u[n] \tag{32}$$

Autocorrelation of the Impulse Response
Denote

$$\hat{R}[i] = \sum_{n=0}^{p} \hat{h}[n] \hat{h}[n-i] \tag{33}$$

It can be shown that (see the last slide)

$$\hat{R}[i] = \sum_{k=1}^{p} \gamma_k \hat{R}[|i-k|], \quad \hat{R}[0] = \sum_{k=1}^{p} \gamma_k \hat{R}[k] + G^2 \tag{34}$$

Cepstrum of $\hat{h}[n]$
This is given by (see the last slide)

$$\hat{h}[n] = c_n + \sum_{k=1}^{n-1} \frac{k}{n} \hat{h}[k] c_{n-k}, \quad n \geq 1 \tag{35}$$

PARCOR
k_i, $i = 1, \ldots, p$. See Figure 9.43 and 9.44.

Log Area Ratio

$$g_i = \log \frac{1 - k_i}{1 + k_i}, \quad 1 \leq i \leq p. \tag{36}$$

Line Spectral Pair
From $A(z)$ we can define the reciprocal polynomial

$$\tilde{A}(z) = z^{-(p+1)} A(z^{-1}) = -\alpha_p z^{-1} - \ldots - \alpha_1 z^{-p} + z^{-(p+1)}. \tag{37}$$

The roots of $\tilde{A}(z)$ is the inverse of the roots of $A(z)$. Consider the line spectral pair polynomials

$$\begin{aligned} P(z) &= A(z) + \tilde{A}(z) = A(z) + z^{-(p+1)} A(z^{-1}) \\ Q(z) &= A(z) - \tilde{A}(z) = A(z) - z^{-(p+1)} A(z^{-1}). \end{aligned} \tag{38}$$

Defining

$$F(z) = \frac{\tilde{A}(z)}{A(z)}, \tag{39}$$

which is an all-pass system,

$$|F(e^{j\omega})| = \left| e^{-j(p+1)\omega} \right| \left| \frac{A(e^{-j\omega})}{A(e^{j\omega})} \right| = 1. \tag{40}$$

The roots of $P(z)$ correspond to $F(z) = -1$, and the roots of $Q(z)$ correspond to $F(z) = 1$. On the unit circle we have

$$F(e^{j\omega}) = -1 \quad \Rightarrow \quad \arg[F(e^{j\omega})] = 2\pi k + \pi, \quad k = 0, \ldots, p-1 \tag{41}$$

for $P(e^{j\omega}) = 0$, and

$$F(e^{j\omega}) = 1 \quad \Rightarrow \quad \arg[F(e^{j\omega})] = 2\pi k, \quad k = 0, \ldots, p-1 \tag{42}$$

for $Q(e^{j\omega}) = 0$. These ω_k are called line-spectral frequencies. Figure 9.47 gives an example.