Transform Analysis of Linear Time-Invariant Systems

Discrete-Time Signal Processing

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Introduction

- A linear time-invariant system is completely characterized by the impulse response $h[n]$.
- We can apply certain transforms to $h[n]$, such as the z-transform and the Fourier transform, and get equivalent representations.
- Looking at a system through these transforms is called **transform analysis**.

Frequency Response

- The frequency response $H(e^{j\omega})$ of a system is the eigenvalue for the input function $e^{j\omega n}$.
- The input/output signal's Fourier transforms are related by

$$Y(e^{j\omega}) = X(e^{j\omega})\; H(e^{j\omega}).$$

- Thus, the magnitude and phase of input/output signals are related by

$$|Y(e^{j\omega})| = |X(e^{j\omega})|\; |H(e^{j\omega})|$$
$$\angle Y(e^{j\omega}) = \angle X(e^{j\omega}) + \angle H(e^{j\omega})$$

Magnitude and Phase Response

- $|H(e^{j\omega})|$ is called the **magnitude response** or **gain**.
- $\angle H(e^{j\omega})$ is called the **phase response** or **phase shift**.
- Combined, they determine the system function,

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{\angle H(e^{j\omega})}.$$

- They are sometimes called **distortion.**

Ideal Frequency Selective Filters

- The **ideal lowpass filter** is characterized by the frequency response

$$H_{\mathsf{l}p}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

- The **ideal highpass filter** is characterized by the frequency response

$$H_{\mathsf{h}p}(e^{j\omega}) = \begin{cases} 0, & |\omega| < \omega_c \\ 1, & \omega_c < |\omega| \le \pi \end{cases}$$

Magnitude and Phase

- Ideal lowpass filter

$$|H_{\mathsf{lp}}(e^{j\omega})| = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}, \quad \angle H_{\mathsf{lp}}(e^{j\omega}) = 0$$

- Similar for an ideal highpass filter.

Ideal Delay System

- The impulse response is

$$h_{\mathsf{id}}[n] = \delta[n - n_d].$$

- The frequency response is the Fourier transform

$$H_{\mathsf{id}}(e^{j\omega}) = e^{-j\omega n_d},$$

so

$$|H_{\mathsf{id}}(e^{j\omega})| = 1, \quad \angle H_{\mathsf{id}}(e^{j\omega}) = -\omega n_d.$$

- The phase shift is linear in ω. This is called a **linear-phase** property.

Group Delay

- A measure of the linearity of the phase is the **group delay** defined by

$$\tau(\omega) = \text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}.$$

- Note here $\arg[H(e^{j\omega})]$ is different from $\angle H(e^{j\omega})$. The former is defined to be a **continuous** function of ω.
- An example of analysis is given in Figure 5.1 (system), 5.2 (input) and 5.3 (output).

LCCDE Systems

- One thing about ideal frequency selective filter is that they cannot be implemented with finite computation.
- We need to consider a class of systems that are implementable.
- The class of causal systems characterized by linear constant-coefficient difference equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

is implementable, as the output can be computed recursively.

z-Domain Analysis

- Such a system (described by LCCDE) is best understood through the z-transform.
- Taking the z-transform,

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\Rightarrow \quad H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

General Properties of $H(z)$

- $H(z)$ is a ratio of polynomials in z^{-1}.
- It can be re-written as

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

  - $(1 - c_k z^{-1})$ contributes a zero at $z = c_k$ and a pole at $z = 0$
  - $(1 - d_k z^{-1})$ contributes a pole at $z = c_k$ and a zero at $z = 0$

Example

- Consider the system function

$$H(z) = \frac{(1 + z^{-1})^2}{(1 - \frac{1}{2}z^{-1})(1 + \frac{3}{4}z^{-1})}$$

- Where are the poles and zeros?
- Can you implement this system?

Region of Convergence

- Given a linear difference equation, we can always compute the functional form of $H(z)$.
- However, we cannot decide the **region of convergence**, thus failing to decide $h[n]$.
- The ambiguity can be resolved if further assumptions regarding **stability** and **causality** are given.

Stability and Causality

- Specifically, for a system to be **stable**, the region of convergence must include the unit circle.
- In addition, for a system to be **causal**, the region of convergence must extend to ∞.
- It follows that if a system is causal and stable, *the outmost pole must be inside the unit circle*.

Example

- Consider the LCCDE

$$y[n] - \frac{5}{2}y[n-1] + y[n-2] = x[n].$$

- The system function is

$$H(z) = \frac{1}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})}.$$

- Where are the poles?
- When is the system causal? stable? stable and causal? neither stable nor causal?

Inverse Systems

- The inverse system $H_i(z)$ of a system $H(z)$ is defined to be the system that if it is cascaded with $H(z)$, the overall system is unity.
- That is,

$$H_i(z)H(z) = 1 \;\Rightarrow\; H_i(z) = \frac{1}{H(z)}.$$

- Equivalently

$$h_i[n] * h[n] = \delta[n].$$

Inverse for LCCDE Systems

- Suppose we have an LCCDE system

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

- The inverse has a function form

$$H_i(z) = \left(\frac{a_0}{b_0}\right) \frac{\prod_{k=1}^{N}(1 - d_k z^{-1})}{\prod_{k=1}^{M}(1 - c_k z^{-1})}$$

- The poles of $H(z)$ are the zeros of $H_i(z)$ and vice versa.

Does an Inverse Exist?

- Does an inverse exist for a given system with known $H(z)$ and ROC?
- A necessary condition for an inverse system is that the ROC of $H_i(z)$ must overlap with the ROC of $H(z)$.
- This condition helps to choose the ROC given the functional form of $H_i(z)$.

Example

- Consider the system

$$H(z) = \frac{1 - 0.5z^{-1}}{1 - 0.9z^{-1}},\ \text{ROC } |z| > 0.9.$$

  The pole is 0.9 and the zero is 0.5.

- The inverse system has a system function of

$$H_i(z) = \frac{1 - 0.9z^{-1}}{1 - 0.5z^{-1}},\ \text{ROC } |z| > 0.5,$$

  where ROC is decided such that it overlaps with $|z| > 0.9$.

Another Example

- Consider the system

$$H(z) = \frac{z^{-1} - 0.5}{1 - 0.9z^{-1}}, \text{ ROC } |z| > 0.9.$$

The pole is 0.9 and the zero is 2.

- The inverse system has a system function of

$$H_i(z) = \frac{1 - 0.9z^{-1}}{z^{-1} - 0.5} = \frac{-2 + 1.8z^{-1}}{1 - 2z^{-1}}.$$

Now both $|z| > 2$ and $|z| < 2$ overlaps with $|z| > 0.9$, so we have a choice.

Causal and Stable Inverse

- Suppose a system with system function $H(z)$ has zeros c_k's and poles d_k's. If it is causal and stable,

$$\max_k |d_k| < 1.$$

- An inverse system has poles c_k's. For it to be causal and stable,

$$\max_k |c_k| < 1.$$

- *A causal and stable system has a causal and stable inverse system if and only if all poles and zeros are inside the unit circle*.

Rational Causal System Function

- Any rational function of z^{-1} with only first-order poles can be written as

$$H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}.$$

- Assume the system is causal, the impulse response is

$$h[n] = \sum_{r=0}^{M-N} B_r \delta[n-r] + \sum_{k=1}^{N} A_k d_k^n u[n].$$

IIR and FIR Systems

- If the system has any poles other than $z = 0$, then $A_k \neq 0$ for some k and $h[n]$ is not of finite length.
- The system is called a **infinite impulse response (IIR)** system.
- Otherwise, $A_k = 0$ for all k and $h[n]$ is of finite duration. Such a system is called **finite impulse response (FIR)**.
- Specifically, an FIR system has $N = 0$ and

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}.$$

Example

- Consider the LCCDE

$$y[n] - ay[n-1] = x[n].$$

- The system function is

$$H(z) = \frac{1}{1 - az^{-1}}.$$

- It is IIR.

Frequency Response

- Suppose a causal LCCDE system is stable.
- The frequency response is the system function evaluated on the unit circle

$$H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}$$

- We are interested in the **magnitude, phase** and **group delay** of such a system.

Magnitude Response

- The equation can be re-written as

$$H(e^{j\omega}) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M} (1 - c_k e^{-j\omega})}{\prod_{k=1}^{N} (1 - d_k e^{-j\omega})}$$

- So the magnitude response is

$$|H(e^{j\omega})| = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=1}^{M} |1 - c_k e^{-j\omega}|}{\prod_{k=1}^{N} |1 - d_k e^{-j\omega}|}$$

Gain and Attenuation

- It is common practice to measure the magnitude in **decibels** (dB), which is defined by

$$\text{Gain in dB} = 20 \log_{10} |H(e^{j\omega})|.$$

- When the gain $|H(e^{j\omega})| < 1$, we speak of **attenuation** in dB, which is defined by

$$\text{Attenuation in dB} = -20 \log_{10} |H(e^{j\omega})|.$$

Gain, Phase and Delay

- For a causal stable LCCDE system,

$$\log_{10}|H(e^{j\omega})| = \log_{10}\left|\frac{b_0}{a_0}\right| + \sum_{k=1}^{M}\log_{10}|1-c_k e^{-j\omega}| - \sum_{k=1}^{N}\log_{10}|1-d_k e^{-j\omega}|.$$

- The phase response is

$$\angle H(e^{j\omega}) = \angle\left(\frac{b_0}{a_0}\right) + \sum_{k=1}^{M}\angle(1-c_k e^{-j\omega}) - \sum_{k=1}^{N}\angle(1-d_k e^{-j\omega})$$

- Similar for the group delay.
- Note that poles and zeros contribute to the sum similarly. Only the signs are opposite.

A Single Zero

- Magnitude, phase, and group delay are sums of terms related to a zero or a pole.
- We now look at such a term. Suppose we have a zero at

$$z_0 = re^{j\theta}.$$

- The squared magnitude is

$$|1-re^{j\theta}e^{-j\omega}|^2 = (1-re^{j\theta}e^{-j\omega})(1-re^{-j\theta}e^{j\omega}) = 1+r^2-2r\cos(\omega-\theta).$$

- This is a periodic function of ω, with period 2π.

Example

- The maximum occurs at $\omega - \theta = \pi$, while the minimum occurs at $\omega - \theta = 0$.
- For example, for $r = 0.9$, the maximum value gain in dB is

$$20 \log_{10}(1 + 0.9) = 5.57,$$

and the minimum is

$$20 \log_{10}(1 - 0.9) = -20.$$

- Varying θ simply shifts the function.

Phase and Group Delay

- The phase of such a term is

$$\text{ARG}\left[1 - re^{j\theta}e^{-j\omega}\right] = \arctan\left[\frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)}\right]$$

- The group delay is (need some basic calculus here, exercise)

$$\text{grd}\left[1 - re^{j\theta}e^{-j\omega}\right] = \frac{r^2 - r\cos(\omega-\theta)}{1 + r^2 - 2r\cos(\omega-\theta)}$$

Geometric Construction

- Again, we look at a first-order system

$$H(z) = (1 - re^{j\theta} z^{-1}) = \frac{z - re^{j\theta}}{z}.$$

- The magnitude response is

$$\left| \frac{e^{j\omega} - re^{j\theta}}{e^{j\omega}} \right| = \frac{|\nu_3|}{|\nu_1|} = |\nu_3|.$$

  - ν_3, a vector from a zero to the unit circle, is called a **zero vector**.
  - ν_1, a vector from a pole to the unit circle, is called a **pole vector**.

Geometric Construction

- The corresponding phase is

$$\angle(1 - re^{j\theta}e^{-j\omega}) = \angle\nu_3 - \angle\nu_1 = \angle\nu_3 - \omega.$$

- We should now have a more concrete idea about the variation of magnitude and phase with respect to ω, from the zero-pole plot of $H(z)$.

Examples

- Figure 5.11 plots the frequency response for $\theta = \pi$ and $r = 1, 0.9, 0.7,$ and 0.5.
- For the magnitude, it is maximum when $\omega - \theta = \pi$ and minimum when $\omega - \theta = 0$, agreeing with (a).
- For the phase, the increase of $\angle \nu_3$ is smaller than $\angle \nu_1$ when $0 \leq \omega \leq \pi/2$, and is larger when $\omega \sim \theta$, agreeing with (b).
- For the group delay, $\angle \nu_3$ increases fastest at $\omega \sim \theta$, agreeing with (c).

More Examples

- Figure 5.13 plots the frequency response for $\theta = \pi$ and $r = 1/0.9, 1.25,$ and 2.0. The zero is outside the unit circle.
- The magnitude follows a pattern similar to Figure 5.11.
- For the phase, the increase of $\angle \nu_3$ is smaller than $\angle \nu_1$ when $0 \leq \omega \leq \pi/2$. In contrast to Figure 5.13, $\angle \nu_3$ actually decrease at $\omega \sim \theta$, leading to (b). Note there is a jump from $-\pi$ to π at $\omega = \theta$.
- The group delay is always positive.

Second-Order System

- Consider a system with complex conjugate pole,

$$H(z) = \frac{1}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})}$$

- The magnitude, phase and group delay are

$$-10\log_{10}[1 + r^2 - 2r\cos(\omega - \theta)] - 10\log_{10}[1 + r^2 - 2r\cos(\omega + \theta)],$$

$$-\arctan\left[\frac{r\sin(\omega - \theta)}{1 - r\cos(\omega - \theta)}\right] - \arctan\left[\frac{r\sin(\omega + \theta)}{1 - r\cos(\omega + \theta)}\right],$$

$$-\frac{r^2 - r\cos(\omega - \theta)}{1 + r^2 - 2r\cos(\omega - \theta)} - \frac{r^2 - r\cos(\omega + \theta)}{1 + r^2 - 2r\cos(\omega + \theta)}.$$

- Plots are given in Figure 5.16.

Magnitude and Phase

- In general, the magnitude response and phase response of a system are independent.
- For an LCCDE system, there is some constraint between them.
- Under a condition referred to as **minimum phase**, the magnitude completely determines the phase and vice versa.

Given Magnitude Response

- Suppose only the magnitude response is given for a system. That is, we have

$$|H(e^{j\omega})|^2 = H(e^{j\omega})H^*(e^{j\omega}) = H(z)H^*(1/z^*)|_{z=e^{j\omega}}.$$

- For a causal system characterized by LCCDE,

$$H(z) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})},\ H^*\left(\frac{1}{z^*}\right) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}(1-c_k^* z)}{\prod_{k=1}^{N}(1-d_k^* z)}.$$

- What is the choice of the system function $H(z)$?

$C(z)$

- $|H(e^{j\omega})|^2$ is the following function evaluated at the unit circle

$$C(z) = H(z)H^*(1/z^*) = \left(\frac{b_0}{a_0}\right)^2 \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})(1 - c_k^* z)}{\prod_{k=1}^{N}(1 - d_k z^{-1})(1 - d_k^* z)}$$

  - For each pole d_k of $H(z)$, there is a pole of $C(z)$ at d_k and $(d_k^*)^{-1}$.
  - For each zero c_k of $H(z)$, there is a zero of $C(z)$ at c_k and $(c_k^*)^{-1}$.
- $C(z)$ can be obtained by replacing $e^{j\omega}$ by z in $|H(e^{j\omega})|^2$.

Poles and Zeros

- The poles of $C(z)$ occur in conjugate reciprocal pairs, i.e., pairs of d_k and $(d_k^*)^{-1}$.
  - If one is inside the unit circle, the other is outside.
  - Or they coincide on the unit circle.
- Similarly for the zeros.
- Suppose we have a causal, stable system. Then we know all the poles are inside the unit circle. There are no restriction for the zeros.
- Example 5.11 gives two distinct causal systems with the same $C(z)$.

Poles and Zeros

- If we further assumes that a_k, b_k are real numbers, then the poles and zeros of $H(z)$ must occur in conjugate pairs or are on the real axis.
- Using this result in Example 5.12, there are still 4 choices for $H(z)$ making the system causal and stable.
- If the number of zeros and poles are not limited, the number of causal and stable systems would also be unlimited.

All-Pass System

- Consider a system function of the form

$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}.$$

- On the unit circle,

$$|H_{ap}(e^{j\omega})| = \left| \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}} \right| = |e^{-j\omega}| \left| \frac{1 - a^* e^{j\omega}}{1 - ae^{-j\omega}} \right| = 1,$$

  since $|e^{-j\omega}| = 1$ and $1 - a^* e^{j\omega}$ is the complex conjugate of $1 - ae^{-j\omega}$.

- Such a system is called an **all-pass** system. Note if a is a pole, $1/a^*$ is a zero.

General All-Pass Systems

- For a general all-pass system with real impulse response, the system function can be written as

$$H_{ap}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})},$$

  where d_k is a real pole, and (e_k, e_k^*) are a pair of conjugate poles.
  - For causal and stable all-pass systems, $|d_k| < 1$ and $|c_k| < 1$.
- The number of poles and zeros for such a system is

$$M = N = 2M_c + M_r.$$

Phase Function

- For a first-order causal all-pass system, with $a = re^{j\theta}$, $r < 1$, the phase function is

$$\angle\left(e^{-j\omega}\frac{1 - re^{j(\omega-\theta)}}{1 - re^{-j(\omega-\theta)}}\right) = \angle\, e^{-j\omega} + \angle(1 - re^{j(\omega-\theta)}) - \angle(1 - re^{-j(\omega-\theta)})$$
$$= -\omega - 2\arctan\left[\frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)}\right].$$

- For a second-order causal all-pass system, with poles at $z = re^{\pm j\theta}$, the phase function is the sum of two similar terms,

$$-2\omega - 2\arctan\left[\frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)}\right] - 2\arctan\left[\frac{r\sin(\omega+\theta)}{1 - r\cos(\omega+\theta)}\right].$$

Phase Property

- The plots of Figure 5.22 and 5.23 are those of a first and second-order causal all-pass systems respectively.
- One can see that in the examples, the phases are non-positive for $0 \le \omega \le \pi$.
- The non-positiveness of phase is a general property of causal all-pass systems.

Group Delay

- The group delay of a first-order causal all-pass system is

$$\text{grd}\left[\frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}\right] = \frac{1 - r^2}{|1 - re^{j\theta}e^{-j\omega}|^2} > 0.$$

- For a general causal all-pass system with real impulse response,

$$\arg H(e^{j0}) = \arg\left(A\prod_{k=1}^{M_r}\frac{1 - d_k}{1 - d_k}\prod_{k=1}^{M_c}\frac{(1 - e_k^*)(1 - e_k)}{(1 - e_k)(1 - e_k^*)}\right) = 0.$$

- Thus the continuous phase $\arg H(e^{j\omega})$ is non-positive.

Minimum-Phase System

- Given $C(z)$, the square of magnitude response of a rational system, $H(z)$ is not uniquely determined.
- We can require the system to be causal and stable, then the poles must be inside the unit circle. Still the system is not unique.
- If we further require the inverse system to be causal and stable, then the zeros must be inside the unit circle. Now the system is unique.
- The **minimum-phase system** is defined to be such a system.

Decomposition

- Suppose $H(z)$ has all zeros and poles are inside the unit circle except a zero at $z = 1/c^*$. Re-write

$$H(z) = H'(z)(z^{-1} - c^*) = H'(z)(1 - cz^{-1})\frac{z^{-1} - c^*}{1 - cz^{-1}}$$

  - $H'(z)(1 - cz^{-1})$ has all the poles and zeros inside the unit circle, thus it is minimum-phase.
  - $(z^{-1} - c^*)/(1 - cz^{-1})$ is causal all-pass.

- By induction, we conclude that any rational system function can be written as

$$H(z) = H_{min}(z)H_{ap}(z).$$

Example

- Consider the system function

$$H(z) = \frac{1 + 3z^{-1}}{1 + \frac{1}{2}z^{-1}}.$$

- The pole is inside the unit circle, but the zero is outside at $z = -3$.
- Identifying $c = -\frac{1}{3}$, we have

$$H(z) = H(z)\frac{1 + \frac{1}{3}z^{-1}}{1 + \frac{1}{3}z^{-1}} = 3\left(\frac{1 + \frac{1}{3}z^{-1}}{1 + \frac{1}{2}z^{-1}}\right)\left(\frac{z^{-1} + \frac{1}{3}}{1 + \frac{1}{3}z^{-1}}\right) = H_{min}(z)H_{ap}(z).$$

Second-Order Example

- Consider the system function

$$H(z) = \frac{(1 + \frac{3}{2}e^{j\pi/4}z^{-1})(1 + \frac{3}{2}e^{-j\pi/4}z^{-1})}{1 - \frac{1}{3}z^{-1}}.$$

- The pole is inside the unit circle, but the zeros are outside at $z = -\frac{3}{2}e^{\pm j\pi/4}$.
- Moving the zeros one by one, we have

$$H(z) = H(z)\frac{1 + \frac{2}{3}e^{j\pi/4}z^{-1}}{1 + \frac{2}{3}e^{j\pi/4}z^{-1}}\frac{1 + \frac{2}{3}e^{-j\pi/4}z^{-1}}{1 + \frac{2}{3}e^{-j\pi/4}z^{-1}}$$

$$= \frac{(1 + \frac{2}{3}e^{j\pi/4}z^{-1})(1 + \frac{2}{3}e^{-j\pi/4}z^{-1})}{1 - \frac{1}{3}z^{-1}}\frac{(1 + \frac{3}{2}e^{j\pi/4}z^{-1})(1 + \frac{3}{2}e^{-j\pi/4}z^{-1})}{(1 + \frac{2}{3}e^{j\pi/4}z^{-1})(1 + \frac{2}{3}e^{-j\pi/4}z^{-1})}.$$

Properties

- The **minimum phase-lag** property

$$H(z) = H_{min}(z)H_{ap}(z) \;\Rightarrow\; \arg H(e^{j\omega}) = \arg H_{min}(e^{j\omega}) + \arg H_{ap}(e^{j\omega})$$

The property follows since a stable and causal all-pass system has negative phase.

- The **minimum group-delay** property

$$\text{grd}[H(e^{j\omega})] = \text{grd}[H_{min}(e^{j\omega})] + \text{grd}[H_{ap}(e^{j\omega})]$$

The property follows since the group delay of a stable and causal all-pass system is positive.

Linear-Phase Systems

- By definition, a linear-phase system has a frequency response of the following form

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{-j\omega\alpha}.$$

- The phase, and group delay are

$$\angle H(e^{j\omega}) = -\omega\alpha; \;\; \text{grd}[H(e^{j\omega})] = \alpha.$$

Ideal Delay System

- Consider the **ideal delay system** $y[n] = x[n-\alpha]$. The frequency response is

$$H_{id}(e^{j\omega}) = e^{-j\omega\alpha}, \quad |\omega| < \pi.$$

- The magnitude, phase, and group delay are

$$|H_{id}(e^{j\omega})| = 1; \;\; \angle H_{id}(e^{j\omega}) = -\omega\alpha; \;\; \text{grd}[H_{id}(e^{j\omega})] = \alpha.$$

- This is an example of linear-phase systems, with constant magnitude.

Generalized Linear-Phase Systems

- A generalized linear-phase system has a frequency response of the following form,

$$H(e^{j\omega}) = A(e^{j\omega})\; e^{-j\omega\alpha + j\beta},$$

  where α, β are real constants and $A(e^{j\omega})$ is a real function of ω.
- Note that $A(e^{j\omega})$ can be negative, so it is not strictly an amplitude.
- Ignoring any discontinuities, we have a constant group delay

$$\text{grd}[H(e^{j\omega})] = \alpha.$$

Necessary Condition

- For a generalized linear-phase system, we have

$$\begin{aligned} H(e^{j\omega}) &= A(e^{j\omega})\, e^{-j\omega\alpha + j\beta} \\ &= A(e^{j\omega}) \cos(\beta - \omega\alpha) + j A(e^{j\omega}) \sin(\beta - \omega\alpha) \\ &= \sum h[n] e^{-j\omega n} = \sum h[n] \cos \omega n - j \sum h[n] \sin \omega n \\ \Rightarrow \quad & \frac{\sin(\beta - \omega\alpha)}{\cos(\beta - \omega\alpha)} = -\frac{\sum h[n] \sin \omega n}{\sum h[n] \cos \omega n} \\ \Rightarrow \quad & \sum h[n] (\cos(\beta - \omega\alpha) \sin \omega n + \sin(\beta - \omega\alpha) \cos \omega n) = 0 \end{aligned}$$

- It follows that

$$\sum_n h[n] \sin[\omega(n - \alpha) + \beta] = 0 \text{ for all } \omega.$$

Symmetric Cases

- In symmetric cases, we have $h[M-n] = h[n]$ (symmetric with respect to the middle point $\frac{M}{2}$).
- We have the flexibility to choose α, β, but the necessary condition must be satisfied.
- Setting $\beta = 0$ (or π), and 2α to be an integer M, we have

$$\sum_n h[n]\sin[\omega(n-\alpha)+\beta] = \sum_n h[n]\sin[\omega(n-\alpha)]$$

$$= \left(\sum_{n=-\infty}^{\lfloor\frac{M-1}{2}\rfloor} + \sum_{n=\lfloor\frac{M-1}{2}\rfloor+1}^{\lceil\frac{M+1}{2}\rceil-1} + \sum_{n=\lceil\frac{M+1}{2}\rceil}^{\infty} \right) h[n]\sin[\omega(n-\alpha)]$$

$$= \sum_{n=-\infty}^{\lfloor\frac{M-1}{2}\rfloor} (h[n]-h[M-n])\sin[\omega(n-\alpha)] = 0.$$

Anti-Symmetric Cases

- In anti-symmetric cases, we have $h[M-n] = -h[n]$ (anti-symmetric with respect to the middle point $\frac{M}{2}$).
- With the conditions $\beta = \pi/2$, $2\alpha = M$, we have

$$\sum_n h[n] \sin[\omega(n-\alpha)+\beta] = \sum_n h[n] \cos[\omega(n-\alpha)]$$

$$= \left(\sum_{n=-\infty}^{\lfloor \frac{M-1}{2} \rfloor} + \sum_{n=\lfloor \frac{M-1}{2} \rfloor + 1}^{\lceil \frac{M+1}{2} \rceil - 1} + \sum_{n=\lceil \frac{M+1}{2} \rceil}^{\infty} \right) h[n] \cos[\omega(n-\alpha)]$$

$$= \sum_{n=-\infty}^{\lfloor \frac{M-1}{2} \rfloor} (h[n] + h[M-n]) \cos[\omega(n-\alpha)] = 0.$$

- Similarly for $\beta = 3\pi/2$.

Causal GLP Systems

- If the system is causal,

$$\sum_{n=0}^{\infty} h[n] \sin[\omega(n-\alpha)+\beta] = 0 \text{ for all } \omega.$$

- Furthermore, the condition of symmetry and anti-symmetry implies that

$$h[n] = 0, \quad n < 0 \text{ or } n > M.$$

- So it is FIR.

Type I Systems

- A type I system is defined by

$$h[n] = \begin{cases} h[M-n], & 0 \le n \le M \\ 0, & \text{otherwise}, \end{cases} \qquad M \text{ is even.}$$

- The frequency response is

$$H(e^{j\omega}) = \left(\sum_{k=0}^{M/2} a[k] \cos \omega k \right) e^{-j\omega M/2},$$

where

$$a[0] = h[M/2],\ a[k] = 2h[M/2 - k].$$

Frequency Response

- Let $\alpha = M/2$.

$$
\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} = \left(\sum_{n=0}^{\alpha-1} + \sum_{n=\alpha+1}^{M}\right) h[n]e^{-j\omega n} + h[\alpha]e^{-j\omega\alpha} \\
&= \sum_{k=1}^{\alpha} \left(h[\alpha-k]e^{-j\omega(\alpha-k)} + h[\alpha+k]e^{-j\omega(\alpha+k)}\right) + h[\alpha]e^{-j\omega\alpha} \\
&= \sum_{k=1}^{\alpha} h[\alpha-k]\left(e^{-j\omega(\alpha-k)} + e^{-j\omega(\alpha+k)}\right) + h[\alpha]e^{-j\omega\alpha} \\
&= \sum_{k=1}^{\alpha} h[\alpha-k]e^{-j\omega\alpha}\left(e^{j\omega k} + e^{-j\omega k}\right) + h[\alpha]e^{-j\omega\alpha} \\
&= \left(\sum_{k=0}^{\alpha} a[k]\cos\omega k\right) e^{-j\omega\alpha}
\end{aligned}
$$

Type II Systems

- A type II system is defined by

$$h[n] = \begin{cases} h[M-n], & 0 \le n \le M \\ 0, & \text{otherwise,} \end{cases} \qquad M \text{ is odd.}$$

- The frequency response is

$$H(e^{j\omega}) = \left(\sum_{k=1}^{(M+1)/2} b[k] \cos\left[\omega(k - \frac{1}{2}) \right] \right) e^{-j\omega M/2},$$

where

$$b[k] = 2h[(M+1)/2 - k].$$

Frequency Response

- Let $\alpha = M/2$.

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \left(\sum_{n=0}^{\alpha-1/2} + \sum_{n=\alpha+1/2}^{M}\right) h[n]e^{-j\omega n}$$

$$= \sum_{k=1}^{\alpha+1/2} \left(h[\alpha + \frac{1}{2} - k]e^{-j\omega(\alpha+\frac{1}{2}-k)} + h[\alpha - \frac{1}{2} + k]e^{-j\omega(\alpha-\frac{1}{2}+k)}\right)$$

$$= \sum_{k=1}^{\alpha+1/2} h[\alpha + \frac{1}{2} - k] \left(e^{-j\omega(\alpha+\frac{1}{2}-k)} + e^{-j\omega(\alpha-\frac{1}{2}+k)}\right)$$

$$= \sum_{k=1}^{\alpha+1/2} h[\alpha + \frac{1}{2} - k]e^{-j\omega\alpha} \left(e^{j\omega(k-\frac{1}{2})} + e^{-j\omega(k-\frac{1}{2})}\right)$$

$$= e^{-j\omega M/2} \sum_{k=1}^{\alpha+1/2} 2h[\alpha + \frac{1}{2} - k] \cos\left[\omega(k - \frac{1}{2})\right]$$

Type III Systems

- A type III system is defined by

$$h[n] = \begin{cases} -h[M-n], & 0 \le n \le M \\ 0, & \text{otherwise}, \end{cases} \qquad M \text{ is even.}$$

- The frequency response is

$$H(e^{j\omega}) = j \left(\sum_{k=1}^{M/2} c[k] \sin \omega k \right) e^{-j\omega M/2},$$

where

$$c[k] = 2h[M/2 - k].$$

Frequency Response

- Let $\alpha = M/2$. $h[\alpha] = 0$ by definition.

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \left(\sum_{n=0}^{\alpha-1} + \sum_{n=\alpha+1}^{M}\right) h[n]e^{-j\omega n}$$

$$= \sum_{k=1}^{\alpha} \left(h[\alpha - k]e^{-j\omega(\alpha-k)} + h[\alpha + k]e^{-j\omega(\alpha+k)}\right)$$

$$= \sum_{k=1}^{\alpha} h[\alpha - k] \left(e^{-j\omega(\alpha-k)} - e^{-j\omega(\alpha+k)}\right)$$

$$= \sum_{k=1}^{\alpha} h[\alpha - k]e^{-j\omega\alpha} \left(e^{j\omega k} - e^{-j\omega k}\right)$$

$$= je^{-j\omega\alpha} \left(\sum_{k=1}^{\alpha} c[k] \sin \omega k\right)$$

Type IV Systems

- A type IV system is defined by

$$h[n] = \begin{cases} -h[M-n], & 0 \le n \le M \\ 0, & \text{otherwise}, \end{cases} \quad M \text{ is dd.}$$

- The frequency response is

$$H(e^{j\omega}) = j\left(\sum_{k=1}^{(M+1)/2} d[k]\sin\left[\omega(k-\frac{1}{2})\right]\right) e^{-j\omega M/2},$$

where

$$d[k] = 2h[(M+1)/2 - k].$$

Frequency Response

- Let $\alpha = M/2$.

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \left(\sum_{n=0}^{\alpha-1/2} + \sum_{n=\alpha+1/2}^{M}\right) h[n]e^{-j\omega n}$$

$$= \sum_{k=1}^{\alpha+1/2} \left(h[\alpha + \frac{1}{2} - k]e^{-j\omega(\alpha+\frac{1}{2}-k)} + h[\alpha - \frac{1}{2} + k]e^{-j\omega(\alpha-\frac{1}{2}+k)}\right)$$

$$= \sum_{k=1}^{\alpha+1/2} h[\alpha + \frac{1}{2} - k] \left(e^{-j\omega(\alpha+\frac{1}{2}-k)} - e^{-j\omega(\alpha-\frac{1}{2}+k)}\right)$$

$$= \sum_{k=1}^{\alpha+1/2} h[\alpha + \frac{1}{2} - k]e^{-j\omega\alpha} \left(e^{j\omega(k-\frac{1}{2})} - e^{-j\omega(k-\frac{1}{2})}\right)$$

$$= je^{-j\omega M/2} \sum_{k=1}^{\alpha+1/2} 2h[\alpha + \frac{1}{2} - k] \sin\left[\omega(k - 1/2)\right]$$

Example

- Figure 5.36 gives FIR linear-phase systems of each type.
- The frequency responses are given in Examples 5.17-5.20.

$$\begin{cases} H_{\mathsf{I}}(e^{j\omega}) = e^{-j\omega 2}\frac{\sin(5\omega/2)}{\sin(\omega/2)}, & (M=4) \\ H_{\mathsf{II}}(e^{j\omega}) = e^{-j\omega 5/2}\frac{\sin(3\omega)}{\sin(\omega/2)}, & (M=5) \\ H_{\mathsf{III}}(e^{j\omega}) = j[2\sin(\omega)]e^{-j\omega}, & (M=2) \\ H_{\mathsf{IV}}(e^{j\omega}) = j[2\sin(\omega/2)]e^{-j\omega/2}, & (M=1) \end{cases}$$

System Function

- Consider the system function of an FIR linear-phase system

$$H(z) = \sum_{n=0}^{M} h[n] z^{-n}.$$

- For either the symmetric or the anti-symmetric cases,

$$H(z) = \sum_{n=0}^{M} \pm h[M-n] z^{-n} = \sum_{k=0}^{M} \pm h[k] z^{k} z^{-M} = \pm z^{-M} H(z^{-1}).$$

- It follows that if z_0 is a zero, so is z_0^{-1}, since

$$H(z_0^{-1}) = z_0^{M} H(z_0) = 0.$$

Location of Zeros

- If $h[n]$ is real, the zeros come in complex conjugate pairs.
- Let $z_0 = re^{j\theta}$ be a zero. From our discussion, we see that the quadruple

$$(re^{j\theta}, re^{-j\theta}, r^{-1}e^{j\theta}, r^{-1}e^{-j\theta})$$

  are all zeros.
  - $r = 1, \theta \neq 0, \pi$, two zeros;
  - $r \neq 1, \theta = 0, \pi$, two zeros;
  - $r = \pm 1$, one zero;

Symmetric/Anti-symmetric Cases

- For the symmetric cases (Types I and II),

$$H(z) = z^{-M} H(z^{-1}).$$

  - $z = -1$ must be a zero for odd M.

- For the anti-symmetric cases (Types III and IV),

$$H(z) = -z^{-M} H(z^{-1}).$$

  - $z = 1$ must be a zero.
  - $z = -1$ must be a zero for even M.