Basics - II

Contents

- Discrete-time signals
- Discrete-time systems
- Representations of systems
- Fourier analysis
- Sampling
- Random signal

Impulse Response

- The impulse response function of a system is the output when the input is the impulse sequence, i.e.,

$$h[n] = T\{\delta[n]\}. \tag{1}$$

- What are the impulse responses of the accumulator and the moving average?

Linear Time-Invariant Systems

- A discrete-time system T is called linear time-invariant (LTI) if it is linear and time-invariant.
- LTI systems are the most important kind in DSP!

LTI and Convolution

- Suppose T is LTI. Using the linearity and time-invariance of T, we have

$$\begin{aligned} y[n] &= T\{x[n]\} = T\left\{\sum_m x[m]\delta[n-m]\right\} = \sum_m x[m]T\{\delta[n-m]\} \\ &= \sum_m x[m]h[n-m]. \end{aligned} \tag{2}$$

- The output is the convolution of the input sequence and the system's impulse response!

Stability of LTI Systems

- An LTI system T with impulse response $h[n]$ is stable if and only if

$$\sum_n |h[n]| < \infty. \tag{3}$$

- We prove the "only if" part. Suppose T is stable. Consider the input

$$x[n] = \begin{cases} \frac{h[-n]}{|h[-n]|}, & h[-n] \neq 0 \\ 0, & h[-k] = 0. \end{cases} \tag{4}$$

The output has to be bounded. In particular,

$$|y[0]| \leq B \Rightarrow \left|\sum_n x[-n]h[n]\right| \leq \left|\sum_n \frac{h[n]}{|h[n]|}h[n]\right| = \sum_n |h[n]| \leq B \tag{5}$$

Causality of LTI Systems

- An LTI system with impulse response $h[n]$ is causal if and only if

$$h[n] = 0, \quad \text{for } n < 0. \tag{6}$$

- This can be easily seen by

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=0}^{\infty} h[k]x[n-k]. \tag{7}$$

- A sequence $x[n]$ is said to be causal if

$$x[n] = 0, \quad \text{for } n < 0. \tag{8}$$

Examples of Impulse Responses

- ideal delay

$$h[n] = \delta[n - n_d] \tag{9}$$

- moving average

$$h[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_2}^{M_1} \delta[n-k] \tag{10}$$

- accumulator

$$h[n] = \sum_{k=-\infty}^{n} \delta[k] = u[n] \tag{11}$$

- forward difference

$$h[n] = \delta[n+1] - \delta[n] \tag{12}$$

- backward difference

$$h[n] = \delta[n] - \delta[n-1] \tag{13}$$

Connection of Systems

- Suppose we have two LTI systems T_1, T_2, with impulse responses $h_1[n]$, $h_2[n]$ respectively.
- There are two fundamental ways to connect two LTI systems
  - cascade

$$h[n] = h_1[n] * h_2[n]. \tag{14}$$

  - parallel

$$h[n] = h_1[n] + h_2[n]. \tag{15}$$

Examples

- The cascade of a forward difference system and a one-sample delay

$$h[n] = (\delta[n+1] - \delta[n]) * \delta[n-1] = \delta[n] - \delta[n-1] \tag{16}$$

- The cascade of a backward difference system and an accumulator

$$h[n] = u[n] * (\delta[n] - \delta[n-1]) = u[n] - u[n-1] = \delta[n]. \tag{17}$$

Inverse Systems

- The last example illustrates the idea of inverse system.
- Generally, the impulse responses of a system and its inverse system satisfies

$$h[n] * h_i[n] = h_i[n] * h[n] = \delta[n]. \tag{18}$$

- Given $h[n]$, it is difficult to solve for $h_i[n]$ directly.
- The same problem will become much easier after we introduce the z-transform!

Linear Time-Invariant Equations

- A linear constant-coefficient difference equation (LCCDE) characterizes the input-output relation of a system

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]. \tag{19}$$

- N is called the order of this difference equation.
- Some of the systems have been defined by LCCDEs. Others can be converted to LCCDEs.

Example

accumulator

- The accumulator can be re-written as

$$y[n] - y[n-1] = \sum_{k=-\infty}^{n} x[k] - \sum_{k=-\infty}^{n-1} x[k] = x[n]. \tag{20}$$

- It is an LCCDE with

$$N = 1, \ a_0 = 1, \ a_1 = -1, \ M = 0, \ b_0 = 1. \tag{21}$$

- One can also see a recursive representation for $y[n]$

$$y[n] = x[n] + y[n-1]. \tag{22}$$

causal moving average

- The causal MA system is defined by

$$y[n] = \frac{1}{M_1 + 1} \sum_{m=0}^{M_1} x[n-m]. \tag{23}$$

- It is an LCCDE with

$$N = 0, \ a_0 = 1, \ M = M_2, \ b_m = 1/(M_2 + 1). \tag{24}$$

LCCDE to find Inverse

- For the same causal moving average system, we have

$$y[n] - y[n-1] = \frac{1}{M_2 + 1}(x[n] - x[n - M_2 - 1]). \tag{25}$$

- The LCCDE is different! Specifically,

$$N = 1, \ a_0 = 1, \ a_1 = -1, \ M = M_2 + 1, \ b_0 = -b_{M_2+1} = \frac{1}{M_2 + 1}. \tag{26}$$

IIR and FIR

- If $N = 0$ in an LCCDE,

$$y[n] = \sum_{m=0}^{M} \frac{b_m}{a_0} x[n-m]. \tag{27}$$

- The impulse response is of finite length

$$y[n] = h[n] = \sum_{m=0}^{M} \left(\frac{b_m}{a_0}\right) \delta[n-m] = \begin{cases} \left(\frac{b_n}{a_0}\right), & 0 \leq m \leq M \\ 0, & \text{otherwise} \end{cases} \tag{28}$$

- Such a system is called finite impulse response (FIR).
- A system is called infinite impulse response (IIR) if it is not FIR.

Eigenvalues of LTI Systems

- A signal $x[n]$ is an eigenvector of a system T if the output $y[n]$ is a multiple of the input $x[n]$.
- The multiplier $T\{x[n]\} = \lambda x[n]$ is called the eigenvalue for $x[n]$.
- We will show that the complex exponential sequences are the eigenvectors of any LTI systems.
- If $e^{j\omega}$ is input to an LTI system with impulse response $h[n]$, the output is

$$y[n] = x[n] * h[n] = \sum_m h[m]e^{j(n-m)} = \left(\sum_m h[m]e^{-j\omega m}\right) e^{j\omega} = H(e^{j\omega})x[n]. \tag{29}$$

- Therefore, $e^{j\omega}$ is an eigenvector with the eigenvalue of

$$H(e^{j\omega}) = \sum_m h[m]e^{-j\omega m}. \tag{30}$$

Frequency Response

- $H(e^{j\omega})$ is called frequency response of the system.
- In general, $H(e^{j\omega})$ is complex, and we can write it in the *polar form*

$$H(e^{j\omega}) = H_{\mathrm{re}}(e^{j\omega}) + jH_{\mathrm{im}}(e^{j\omega}) = |H(e^{j\omega})|e^{j\arg\{H(e^{j\omega})\}} \tag{31}$$

  - $|H(e^{j\omega})|$ is called the magnitude response
  - $\arg\{H(e^{j\omega})\}$ is called the phase response

Discrete-Time Fourier Transform

- The discrete-time Fourier transform (DTFT) of a sequence $x[n]$ is defined by

$$X(e^{j\omega}) = \sum_n x[n]e^{-j\omega n}. \tag{32}$$

- $X(e^{j\omega})$ is called to as the spectrum of $x[n]$.
- We have shown that the impulse response and the frequency response of an LTI system are related by

$$H(e^{j\omega}) = \sum_m h[m]e^{-j\omega m}. \tag{33}$$

which is the DTFT of $h[n]$.

Frequency Response of Ideal Delay

- The *ideal delay* system has an impulse response of

$$h[n] = \delta[n - n_d], \tag{34}$$

- The frequency response is the DTFT of $h[n]$,

$$H(e^{j\omega}) = \sum_n \delta[n - n_d]e^{-j\omega n} = e^{-j\omega n_d}. \tag{35}$$

- The magnitude and phase responses of $H(e^{j\omega})$ are 1 and $-\omega n_d$, respectively.

Sinusoidal Components

- A broad class of signals can be represented as a linear combination of complex exponentials

$$x[n] = \sum_k \alpha_k e^{j\omega_k n}, \quad \text{where } \omega_k = \frac{2\pi}{N}k. \tag{36}$$

- From the principle of superposition, the output of an LTI system with $x[n]$ as input is simply

$$y[n] = \sum_k \alpha_k e^{j\omega_k n} H(e^{j\omega_k}). \tag{37}$$

Real Frequency and Low Frequency

- We can see that $H(e^{j\omega})$ is always periodic with period 2π.
- Thus, we only need to specify $H(e^{j\omega})$ in a range of length 2π.
- The frequencies near $\omega = 0$ are the "low frequencies", and those near $\omega = \pm\pi$ are the "high frequencies".

Response to a Sinusoidal Input

- Consider the sinusoidal input to an LTI

$$\begin{aligned} x[n] &= A\cos(\omega_0 n + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n} \\ \Rightarrow y[n] &= H(e^{j\omega_0})\frac{A}{2}e^{j\phi}e^{j\omega_0 n} + H(e^{-j\omega_0})\frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}. \end{aligned} \tag{38}$$

- If $h[n]$ is real, $H(e^{-j\omega}) = H^*(e^{j\omega})$, and we have

$$y[n] = A|H(e^{j\omega_0})|\cos(\omega_0 n + \phi + \theta), \quad \text{where } \theta = \arg H(e^{j\omega_0}) \tag{39}$$

- For the ideal delay system, as expected,

$$y[n] = A\cos(\omega_0(n - n_d) + \phi) = x[n - n_d]. \tag{40}$$

Example: Sinusoidal Input

- The impulse response is

$$h[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} \delta[n-k] \tag{41}$$

- The frequency response is

$$\begin{aligned} H(e^{j\omega}) &= \frac{1}{M_1 + M_2 + 1} \sum_{n=-M_1}^{M_2} e^{-j\omega k} \\ &= \frac{1}{M_1 + M_2 + 1} \frac{\sin[\omega(M_1 + M_2 + 1)/2]}{\sin(\omega/2)} e^{-j\omega(M_2 - M_1)/2} \end{aligned} \tag{42}$$

- See Figure 2.19.

Inverse Discrete-Time Fourier Transformation

- $x[n]$ can be reconstructed from $X(e^{j\omega})$, which is called the inverse DTFT

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega. \tag{43}$$

- (43) can be proved by noting that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(n-n')\omega}d\omega = \delta[n-n']. \tag{44}$$

Analysis and Synthesis

- Analysis (DTFT)

$$X(e^{j\omega}) = \sum_n x[n]e^{-j\omega n}. \tag{45}$$

- Synthesis (inverse DTFT)

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega. \tag{46}$$

Properties of DTFT

- Linearity
- Time shifting

$$x[n - n_d] \longleftrightarrow e^{-j\omega n_d}X(e^{j\omega}) \tag{47}$$

- Frequency shifting

$$e^{j\omega_0 n}x[n] \longleftrightarrow X(e^{j(\omega-\omega_0)}) \tag{48}$$

- Time reversal

$$x[-n] \longleftrightarrow X(e^{-j\omega}) \tag{49}$$

- Differentiation

$$nx[n] \longleftrightarrow j\frac{dX(e^{j\omega})}{d\omega} \tag{50}$$

- Parseval's theorem

$$\sum_n |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega \tag{51}$$

- Convolution theorem

$$h[n] * x[n] \longleftrightarrow H(e^{j\omega})X(e^{j\omega}) \tag{52}$$

- Modulation theorem

$$h[n]x[n] \longleftrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})H(e^{j(\omega-\theta)})d\theta \tag{53}$$