Discrete-Time Signals and Systems

Discrete-Time Signal Processing

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Outline

- discrete-time signals
- discrete-time systems
- linear time-invariant systems
- linear difference equations
- frequency-domain representation of signals and systems
- Fourier transform: representation of sequences
- Fourier transform theorems

Discrete-Time Signals

- a discrete-time signal $=$ a **sequence** of numbers

$$x = \{x[n]\}, \ \ -\infty < n < \infty.$$

- For instance, $x[n]$ often arises from **periodic sampling** of a continuous-time signal,

$$x[n] = x_a(nT), \ \ -\infty < n < \infty.$$

  - T: **sampling period**
  - $\frac{1}{T} = f_s$: **sampling frequency**

Basic Sequences

- **impulse sequence**, aka **unit sample sequence**

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

- **unit step sequence**

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

- **sinusoidal sequence**

$$x[n] = A\cos(\omega n + \phi)$$

Basic Sequence Operations

- **shift** or **delay**: $y[n] = x[n - n_0]$ is a shifted or delayed version of $x[n]$ by n_0
- **sum**: the sum of two sequences $x[n], y[n]$ is another sequence

$$z[n] = x[n] + y[n]$$

- **product**: the product of two sequences $x[n], y[n]$ is another sequence

$$z[n] = x[n]\ y[n]$$

- **scaling** $x[n]$ by a factor of α

$$z[n] = \alpha x[n]$$

Decomposition of a Sequence

- Any discrete-time signal can be represented as a sum of **delayed** and **scaled** impulse sequences.
- Specifically

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

- Note we can also write

$$x[n] = \sum_{k=-\infty}^{\infty} x[n-k]\delta[k].$$

Impulse and Unit Step Sequences

- For the unit step sequence

$$
\begin{aligned}
u[n] &= \sum_{k=-\infty}^{\infty} u[k]\delta[n-k] = \sum_{k=0}^{\infty}\delta[n-k] \\
&= \sum_{k=-\infty}^{\infty} u[n-k]\delta[k] = \sum_{k=-\infty}^{n} \delta[k] \\
&= \delta[n] + \delta[n-1] + \ldots
\end{aligned}
$$

- It follows that

$$\delta[n] = u[n] - u[n-1].$$

Exponential Sequences

- An **exponential sequence** is given by

$$x[n] = C\alpha^n.$$

- We can combine with the unit step function such that $x[n] = 0$ for $n < 0$, i.e.,

$$x[n] = C\alpha^n u[n].$$

- C and α are complex numbers, so we can write

$$C = Ae^{j\phi}, \quad \alpha = |\alpha|e^{j\omega_0},$$

where A, ϕ, ω_0 are real numbers.

Sinusoidal Sequences

- A sinusoidal sequence has the form

$$x[n] = A\cos(\omega_0 n + \phi),$$

where A, ω_0 and ϕ are real.

- Note the real and imaginary parts of an exponential sequence are

$$\begin{aligned} x[n] = C\alpha^n = Ae^{j\phi}|\alpha|^n e^{j\omega_0 n} \\ = A|\alpha|^n e^{j\omega_0 n+\phi} \\ = A|\alpha|^n \cos(\omega_0 n + \phi) + jA|\alpha|^n \sin(\omega_0 n + \phi) \end{aligned}$$

Complex Exponential Sequences

- By definition, a **complex exponential** is an exponential sequence with $|\alpha| = 1$, i.e.,

$$x[n] = A\cos(\omega_0 n + \phi) + jA\sin(\omega_0 n + \phi)$$

  - ω_0 is called **frequency**
  - ϕ is called **phase**

- The real and imaginary parts are both sinusoidal sequences. Note that

$$\omega_0, \;\; \omega_r = \omega_0 + 2\pi r, \; r \in Z$$

  are indistinguishable frequencies, since they give identical complex exponential sequences.

Periodicity

- For a given sinusoidal sequence $x[n] = A\cos(\omega_0 n + \phi)$ to be periodic, it is *required* that

$$2\pi k = \omega_0 \Delta n, \quad \text{for some integers } k, \Delta n.$$

- Therefore, a sinusoidal sequence is *not* always periodic in the index n.
- Note this contrasts the continuous-time case, where $x(t) = A\cos(\omega_0 t + \phi)$ is always periodic with period $\frac{2\pi}{\omega_0}$.
- Increasing the frequency may increase the period!

$$x_1[n] = \cos(\frac{\pi}{4}n), \; x_2[n] = \cos(\frac{3\pi}{8}n)$$

Periodic Frequencies

- A discrete-time sinusoidal sequence is N-periodic if $\omega = \frac{2\pi k}{N},\ k \in Z$, since

$$A\cos(\omega(n+N)+\phi) = A\cos(\omega n+\phi).$$

- For a given N, there are N *distinguishable* frequencies for periodic sinusoidal sequences,

$$\omega_k = \frac{2\pi}{N}k,\ k = 0, 1, \ldots, N-1.$$

- Any sequence periodic with N is a linear combination of sinusoidal sequences of these frequencies.

High and Low Frequencies

- The oscillation of a sinusoidal sequence $x[n] = \cos(\omega n)$ does not always increase with ω!
  - $x[n]$ does oscillate more and more rapidly as ω increases from 0 to π
  - oscillation slows down as ω increases from π to 2π

$$\cos(\omega n) = \cos(-\omega n) = \cos((2\pi - \omega)n).$$

- For discrete-time signals, the frequencies near $\omega = 2\pi k$ are the **low** frequencies, while the frequencies near $\omega = \pi + 2\pi k$ are the **high** frequencies.

Discrete-Time Systems

- A discrete-time system is a **transformation** T that maps an input sequence $\{x[n]\}$ to an output sequence $\{y[n]\}$

$$y[n] = T\{x[n]\}.$$

- T is characterized by the exact mathematical formula relating $y[n]$ and $x[n]$.

Examples

- **ideal delay**

$$y[n] = x[n - n_d]$$

- **moving average**

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n - k]$$

Memoryless Systems

- A system is said to be **memoryless** if the value of $y[n]$ at n depends only on the value of $x[n]$ at n.
- For example,

$$y[n] = (x[n])^2.$$

- It does not depend on any earlier value of $x[n]$.

Linear Systems

- A system is said to be **linear** if

$$T\{x_1[n] + x_2[n]\} = T\{x_1[n]\}) + T(\{x_2[n]\}$$
$$T\{ax[n]\} = aT\{x[n]\}$$

- For example, the **accumulator** defined by

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

is linear.

Time-Invariant Systems

- A system is said to be **time-invariant** if

$$y_{n_d}[n] = T\{x[n - n_d]\} = y[n - n_d].$$

- For example, the accumulator is time-invariant, since

$$T\{x[n - n_d]\} = \sum_{k=-\infty}^{n} x[k - n_d] = \sum_{k'=-\infty}^{n-n_d} x[k'] = y[n - n_d].$$

Example: Compressor

- A **compressor** is defined by

$$y[n] = x[Mn], \ -\infty < n < \infty, \ M \in Z^+.$$

- It “compacts” every other M samples of $x[n]$.
- To see that this system is *not* time-invariant, note

$$T\{x[n-n_d]\} = x[Mn - n_d] \neq y[n-n_d] = x[M(n-n_d)].$$

- A counterexample can be established by

$$x[n] = \delta[n], M = 2, n_d = 1$$

$$\Rightarrow y[n] = \delta[2n] = \delta[n] \ \& \ T\{\delta[n-1]\} = 0 \neq y[n-1] = \delta[n-1]$$

Causal Systems

- A system is said to be **causal** if the output value at any n_0 only depends on the input values at $n \leq n_0$.
- In other words, the system is **non-anticipative**.
- Which of the systems are causal
  - accumulator?
  - moving average?

Example: Forward Difference

- A **forward difference** system is defined by

$$y[n] = x[n+1] - x[n].$$

- A **backward difference** system is defined by

$$y[n] = x[n] - x[n-1].$$

- Which is causal? Which is not?

Stability

- A system is said to be **stable** if a bounded input sequence produces a bounded output sequence.

$$|x[n]| \leq B_x < \infty \Rightarrow |y[n]| \leq B_y < \infty$$

- **BIBO**
- Which of the systems are stable
  - accumulator?
  - moving average?

Linear Time-Invariant Systems

- important, important, important
- A linear time-invariant (LTI) system is characterized by its impulse response function.
- basic ideas
  - any input sequence is a linear combination of shifted impulse sequences (**general property**)
  - the output of a shifted impulse is a shifted impulse response (**time-invariance**)
  - the output of any input sequence is the same linear combination of shifted impulse response (**linearity**)

Impulse Response

- The **impulse response function** is the output sequence when the input is the impulse sequence,

$$h[n] = T\{\delta[n]\}.$$

- Suppose a system is LTI. For any input sequence $x[n]$, the output is

$$\begin{aligned} y[n] = T\{x[n]\} &= T\left\{\sum_m x[m]\delta[n-m]\right\} \\ &= \sum_m x[m]T\{\delta[n-m]\} = \sum_m x[m]h[n-m]. \end{aligned}$$

Convolution

- The operation between two sequences

$$x[n] * h[n] = \sum_m x[m]h[n-m]$$

is called the **convolution** of $x[n]$ and $h[n]$, which results in another sequence.
- The summation in an operation of convolution is called **convolution sum**.
- In the derivation, the perspective is to express the resultant sequence as a sum of sequences, i.e., $h[n-m]$.

Evaluation of Convolution

- We now show how to compute $x[n] * h[n]$ at a specific time index n.
- Given n, the convolution sum is the “inner product” of two sequences $x[k]$ and $h[n-k]$ both indexed by k.
  - $x[k]$ is just $x[n]$
  - $h[n-k] = h[-k+n]$ is $h[k]$ *reflected* (with respect to time origin) and then *shifted to the right* by n.

Example

- Let the impulse response and input sequence be

$$x[n] = a^n u[n]; \quad h[n] = u[n] - u[n-N] = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{otherwise}, \end{cases}$$

- The output is

$$y[n] = x[n] * h[n] = \sum_k x[k]h[n-k]$$

$$= \begin{cases} 0, & n < 0, \\ \frac{1-a^{n+1}}{1-a}, & 0 \le n \le N-1 \\ a^{n-N+1}\left(\frac{1-a^N}{1-a}\right), & n > N-1. \end{cases}$$

Convolution Properties

- **commutative**

$$x[n]*h[n] = \sum_m x[m]h[n-m] = \sum_{m'} x[n-m']h[m'] = h[n]*x[n].$$

- **linearity**

$$\begin{aligned} x[n] * (h[n] + g[n]) &= \sum_m x[m](h[n-m] + g[n-m]) \\ &= x[n] * h[n] + x[n] * g[n]. \end{aligned}$$

Connection of Systems

- **cascade connection**: the response to an impulse sequence is

$$h[n] = h_1[n] * h_2[n].$$

- **parallel connection**: the response to an impulse sequence is

$$h[n] = h_1[n] + h_2[n].$$

LTI Causal Systems

- From an LTI system, with impulse response $h[n]$, to be causal, since

$$y[n] = \sum_k h[k]x[n-k],$$

it must hold that

$$h[k] = 0, \quad k < 0.$$

- A sequence $x[n]$ is said to be **causal** if

$$x[n] = 0, \quad n < 0.$$

Examples of Impulse Response

- ideal delay

$$h[n] = \delta[n - n_d]$$

- moving average

$$h[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} \delta[n - k]$$

- accumulator

$$h[n] = u[n]$$

More Examples

- forward difference

$$h[n] = \delta[n+1] - \delta[n]$$

- backward difference

$$h[n] = \delta[n] - \delta[n-1]$$

Equivalent Systems

- The cascade system of a forward difference and a one-sample delay is equivalent to a backward difference system

$$\begin{aligned} h[n] &= (\delta[n+1] - \delta[n]) * \delta[n-1] \\ &= \delta[n] - \delta[n-1] \end{aligned}$$

- The cascade system of a backward difference and a accumulator is equivalent to an identity system

$$\begin{aligned} h[n] &= u[n] * (\delta[n] - \delta[n-1]) = u[n] - u[n-1] \\ &= \delta[n]. \end{aligned}$$

Inverse Systems

- The last example is an example of **inverse system**.
- More generally, the impulse response functions of a system and its inverse system satisfies

$$h[n] * h_i[n] = h_i[n] * h[n] = \delta[n].$$

- Given $h[n]$, it is difficult to solve for $h_i[n]$ directly.
- With z-transform, this problem becomes much easier!

Linear Difference Equations

- a class of system representations
- The input and output sequences are related by a **linear constant-coefficient difference equation (LCCDE)**

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m].$$

- N is said to be the **order** of this difference equation.

Accumulator as LCCDE

- The accumulator can be represented by an LCCDE

$$y[n] - y[n-1] = \sum_{k=-\infty}^{n} x[k] - \sum_{k=-\infty}^{n-1} x[k] = x[n],$$

corresponding to $N = 1, a_0 = 1, a_1 = -1, M = 0, b_0 = 1$.

- One can also see a **recursive representation** for $y[n]$

$$y[n] = x[n] + y[n-1].$$

- offering another picture of a system

Causal Moving Average as LCCDE

- Recall that

$$y[n] = \frac{1}{M_2 + 1} \sum_{m=0}^{M_2} x[n-m].$$

- an LCCDE with $N = 0, a_0 = 1, M = M_2, b_m = 1/(M_2 + 1)$
- We can express the impulse response as

$$h[n] = \frac{1}{M_2 + 1} (\delta[n] - \delta[n - M_2 - 1]) * u[n].$$

- Note how a causal moving average is equivalent to attenuation, ideal delay and accumulator.

More on Causal Moving Average

- If we define

$$x_1[n] = \frac{1}{M_2+1}(x[n] - x[n-M_2-1]),$$

we have

$$y[n] - y[n-1] = x_1[n] = \frac{1}{M_2+1}(x[n] - x[n-M_2-1]).$$

- Note we have another LCCDE for the same system! Specifically,

$$N = 1, a_0 = 1, a_1 = -1,\ M = M_2+1, b_0 = b_{M_2+1} = 1/(M_2+1).$$

Recursive Computation of Solution

- Suppose an LCCDE is given for a system. With input sequence $x[n]$ and initial conditions $y[-1], \ldots, y[-N]$, $y[n]$ can be computed recursively as follows.
  - For $n > 0$, starting from $n = 0$ and recursively,

$$y[n] = -\sum_{k=1}^{N} \frac{a_k}{a_0} y[n-k] + \sum_{m=0}^{M} \frac{b_m}{a_0} x[n-m].$$

  - For $n < -N$, starting from $l = 1$ and recursively,

$$y[-N-l] = -\sum_{k=0}^{N-1} \frac{a_k}{a_N} y[-k-l] + \sum_{m=0}^{M} \frac{b_m}{a_N} x[-m-l].$$

Example

$$\begin{cases} y[n] = ay[n-1] + x[n]; \\ y[-1] = c; \ x[n] = K\delta[n]; \end{cases}$$

- For $n \geq 0$, from $y[-1] = c$ and recursion,

$$y[0] = ac + K, \ y[1] = a(ac + K), \ldots, \ y[n] = a^{n+1}c + a^n K$$

- For $n < -1$, from $y[-1] = c$ and recursion,

$$\begin{aligned} y[-2] &= a^{-1}(y[-1] - x[-1]) = a^{-1}c, \\ y[-3] &= a^{-1}(y[-2] - x[-2]) = a^{-2}c, \\ &\vdots \\ y[n] &= a^{n+1}c \end{aligned}$$

Consistent Conditions

- In the current example, the system is not linear and not time-invariant.
  - not linear: $K = 0$, then $x[n] = 0$ but $y[n] \neq 0$
  - not time invariant: $x'[n] = K\delta[n - n_0]$, then

$$y'[n] = a^{n+1}c + Ka^{n-n_0}u[n - n_0] \neq y[n - n_0].$$

  - not causal: $y[-1] = c \neq 0$
- We have a system described by LCCDE but is not LTI!
- If a system described by LCCDE is required to be LTI and causal, then the solution is unique!
- It must have **initial-rest conditions**: the first non-zero output point cannot precede the first non-zero input.

FIR Filters

- FIR: **finite impulse response**
- In an LCCDE, if $N = 0$, then

$$y[n] = \sum_{m=0}^{M} \frac{b_m}{a_0} x[n-m].$$

- Let $x[n] = \delta[n]$, then

$$y[n] = h[n] = \sum_{m=0}^{M} \left(\frac{b_m}{a_0}\right) \delta[n-m] = \begin{cases} \left(\frac{b_m}{a_0}\right), & 0 \leq m \leq M \\ 0, & \text{otherwise} \end{cases}$$

- This is FIR.

Frequency-Domain Representation

- A discrete-time signal may be represented in a number of different ways.
  - A *periodic* sequence can be represented as a sum of sinusoidal sequences of certain frequencies.
- For LTI systems, **sinusoidal** and **complex exponential** sequences are of particular importance.
- They are the **eigenfunctions** of LTI systems, as we show below.

Eigenvector of LTI System

- If a complex exponential $x[n] = e^{j\omega n}$ is input to an LTI system, with impulse response $h[n]$, the output is

$$y[n] = x[n]*h[n] = \sum_m h[m]e^{j\omega(n-m)} = e^{j\omega n}\sum_m h[m]e^{-j\omega m} = x[n]H(e^{j\omega}).$$

- The output is a multiple of the input!
- The key idea is to see a sequence as a **vector**, and see a system as a **linear transformation**.
- Clearly $e^{j\omega n}$ is an **eigenvector** with **eigenvalue**

$$H(e^{j\omega}) = \sum_m h[m]e^{-j\omega m}.$$

Frequency Response

- $H(e^{j\omega})$ is called the **frequency response** of the system.
- In general, $H(e^{j\omega})$ is complex

$$H(e^{j\omega}) = H_R(e^{j\omega}) + jH_I(e^{j\omega}) = |H(e^{j\omega})|e^{j\arg(H(e^{j\omega}))}$$

  - $|H(e^{j\omega})|$ is called **magnitude response**
  - $\arg(H(e^{j\omega}))$ is called **phase response**

Fundamental Relationship

- The **Fourier transform** of a discrete-time signal $x[n]$ is defined by

$$X(e^{j\omega}) = \sum_{n} x[n]e^{-j\omega n}.$$

- We have just shown that, for an LTI system, *the frequency response and the impulse response are related by the Fourier transform*!

$$H(e^{j\omega}) = \sum_{m} h[m]e^{-j\omega m}.$$

Example: Ideal Delay

- ideal delay
  - via Fourier transform of $h[n]$

$$H(e^{j\omega}) = \sum_n \delta[n - n_d] e^{-j\omega n} = e^{-j\omega n_d}.$$

  - via direct computation: $x[n] = e^{j\omega n}$,

$$y[n] = x[n - n_d] = e^{j\omega(n - n_d)} = e^{j\omega n} e^{-j\omega n_d} = x[n] H(e^{j\omega})$$
$$\Rightarrow \;\; H(e^{j\omega}) = e^{-j\omega n_d}$$

- The magnitude and phase of $H(e^{j\omega})$ is 1 and $-\omega n_d$ respectively.

Decomposition

- Suppose $x[n]$ is a linear combination of $e^{j\omega_k n}$ of different ω_k's

$$x[n] = \sum_k \alpha_k e^{j\omega_k n}.$$

- The output of an LTI system is, by *the principle of superposition*,

$$y[n] = \sum_k \alpha_k e^{j\omega_k n} H(e^{j\omega_k}).$$

- For any input sequence, we just need to figure out the linear combination to compute the output.

Example

- Let $x[n]$ be a sinusoidal sequence

$$x[n] = A\cos(\omega_0 n + \phi).$$

- decomposition by complex exponential sequences

$$A\cos(\omega_0 n + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}.$$

- The output is

$$y[n] = \frac{A}{2}e^{j\phi}H(e^{j\omega_0 n})e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}H(e^{-j\omega_0 n})e^{-j\omega_0 n}$$

Properties of Frequency Response

- For any LTI system, the frequency response is always periodic, with period 2π, since

$$H(e^{j(\omega+2\pi)}) = \sum_n h[n]e^{-j(\omega+2\pi)n} = \sum_n h[n]e^{-j\omega n} = H(e^{j\omega})$$

- This is to be expected as $\{e^{j\omega n}\}$ and $\{e^{j(\omega+2\pi)n}\}$ are identical sequences.
- We only need to specify $H(e^{j\omega})$ over one period, say $[0, 2\pi]$ or $[-\pi, \pi]$.

Frequency Selective Filters

- **ideal low-pass filters**

$$H_{\mathsf{lp}}(e^{j\omega}) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

- **ideal high-pass filters**

$$H_{\mathsf{hp}}(e^{j\omega}) = \begin{cases} 0, & |\omega| \leq \omega_c \\ 1, & \omega_c < |\omega| \leq \pi \end{cases}$$

- **ideal band-pass filters** are similarly defined with cut-off frequencies ω_a, ω_b.

Moving-Average System

- Recall that

$$h[n] = \begin{cases} \frac{1}{M_1+M_2+1}, & -M_1 \le n \le M_2 \\ 0, & \text{otherwise} \end{cases}.$$

- The frequency response is

$$H(e^{j\omega}) = \frac{1}{M_1 + M_2 + 1} \sum_{n=-M_1}^{M_2} e^{-j\omega n}$$
$$= \frac{1}{M_1 + M_2 + 1} \frac{\sin[\omega(M_1 + M_2 + 1)/2]}{\sin(\omega/2)} e^{-j\omega(M_2 - M_1)/2}.$$

(cf 2.19) The low-frequency part is more emphasized.

Causal Complex Exponentials

- A complex exponential that extends to both sides of infinity seems impractical. Instead, consider a suddenly applied exponential

$$x[n] = e^{j\omega n} u[n].$$

- For a causal LTI system, the output is

$$y[n] = \sum h[k] x[n-k] = \begin{cases} 0, & n < 0 \\ \sum_{k=0}^{n} \left(h[k] e^{-j\omega k} \right) e^{j\omega n}, & n \geq 0 \end{cases}$$

State-State and Transient

- For $n \geq 0$, we write

$$y[n] = \left(\sum_{k=0}^{n} h[k]e^{-j\omega k}\right) e^{j\omega n} = H(e^{j\omega})e^{j\omega n} - \left(\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}\right) e^{j\omega n}$$
$$= y_{SS}[n] + y_t[n].$$

- $y_{SS}[n]$ is called the state-state response, given by

$$y_{SS}[n] = H(e^{j\omega})e^{j\omega n}.$$

- $y_t[n]$ is called the transient response, given by

$$y_t[n] = -\left(\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}\right) e^{j\omega n}.$$

Stable System

- Under certain conditions, $y_t[n]$ vanishes as $n \rightarrow \infty$.
- Specifically,

$$|y_t[n]| = \left| \left(\sum_{k=n+1}^{\infty} h[k] e^{-j\omega k} \right) e^{j\omega n} \right| \leq \sum_{k=n+1}^{\infty} |h[k]|$$

- If the system is stable, i.e.,

$$\sum_{k=0}^{\infty} |h[k]| < \infty \Rightarrow \sum_{k=n+1}^{\infty} |h[k]| \rightarrow 0,$$

So $y_t[n] \rightarrow 0$.

Representations of a Sequence

- An LTI system is characterized by its frequency response, the Fourier transform of the impulse response

$$h[n] \leftrightarrow H(e^{j\omega}).$$

- Likewise, a sequence $x[n]$ can be represented by its Fourier transform $X(e^{j\omega})$

$$X(e^{j\omega}) = \sum x[n]e^{-j\omega n}.$$

- Indeed, from $X(e^{j\omega})$ we can reconstruct $x[n]$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega.$$

Analysis and Synthesis

- **Fourier integral = synthesis**: $x[n]$ as a superposition of (infinitesimal) complex exponentials

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

also known as the **inverse Fourier transform**

- **Fourier transform = analysis**: for the the weights of complex exponentials in $x[n]$

$$X(e^{j\omega}) = \sum x[n]e^{-j\omega n}$$

Proof of Inverse

- Plugging in the analysis formula into the rhs of the synthesis formula, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_m x[m]e^{-j\omega m}\right)e^{j\omega n}d\omega = \sum_m x[m]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{j\omega(n-m)}d\omega\right)$$

- Note that

$$\int_{-\pi}^{\pi}e^{j\omega(n-m)}d\omega = \begin{cases}2\pi, & n=m\\ 0, & n\neq m\end{cases}.$$

- So the integral yields $x[n]$.

Convergence

- The infinite sum for the analysis formula has to converge for $X(e^{j\omega})$ to be defined.
- What $x[n]$ has a convergent $X(e^{j\omega})$?
- A sufficient condition is that $x[n]$ is **absolutely summable**, i.e.,

$$\sum_n |x[n]| < \infty \Rightarrow |X(e^{j\omega})| \leq \sum |x[n]| \leq \infty$$

- A **stable** sequence, by definition, is absolutely summable, so it has a Fourier transform.

Square Summable Sequence

- Consider the ideal low-pass filter

$$H_{\text{lp}}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| < \pi \end{cases},$$

  the impulse response is

$$h[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega = \frac{1}{2\pi j n} e^{j\omega n}\Big|_{-\omega_c}^{\omega_c} = \frac{\sin \omega_c n}{\pi n}.$$

- not **absolutely summable**, but **square summable**
- an example of Fourier transform representation for a sequence not absolutely summable

Constant Sequence

- A constant sequence is neither absolutely summable nor square summable,

$$x[n] = 1.$$

- Yet, we can define the Fourier transform to be

$$X(e^{j\omega}) = \sum_r 2\pi\delta(\omega + 2\pi r).$$

- This is justified by the fact that substituting $X(e^{j\omega})$ into the synthesis formula yields $x[n]$.

Complex Exponential

- The FT of a constant sequence is a periodic (2π) impulse train.
- What if we shift the impulse by ω_0, i.e.,

$$X(e^{j\omega}) = \sum_r 2\pi\delta(\omega - \omega_0 + 2\pi r)?$$

- It is the FT of $e^{j\omega_0 n}$ since

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$
$$=\frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\delta(\omega - \omega_0)e^{j\omega n}d\omega = e^{j\omega_0 n}.$$

Sum of Discrete-Frequency Components

- Suppose a sequence is a sum of discrete-frequency exponential components,

$$x[n] = \sum_k a_k e^{j\omega_k n}.$$

- Then the Fourier transform representation is

$$X(e^{j\omega}) = \sum_r \sum_k 2\pi a_k \delta(\omega - \omega_k + 2\pi r).$$

Symmetric/Antisymmetric Sequences

- **conjugate-symmetric** sequence

$$x[n] = x^*[-n].$$

- **conjugate-antisymmetric** sequence

$$x[n] = -x^*[-n].$$

- Any sequence $x[n]$ is a sum of conjugate-symmetric and conjugate-antisymmetric sequences,

$$x[n] = \frac{1}{2}(x[n] + x^*[-n]) + \frac{1}{2}(x[n] - x^*[-n]) = x_s[n] + x_a[n]$$

Real Sequences

- A real sequence is **even** if

$$x[n] = x[-n].$$

- A real sequence is **odd** if

$$x[n] = -x[-n].$$

- Any real sequence $x[n]$ is a sum of even and odd sequences,

$$x[n] = \frac{1}{2}(x[n] + x[-n]) + \frac{1}{2}(x[n] - x[-n]) = x_e[n] + x_o[n]$$

Regarding Fourier Transform

- $X(e^{j\omega})$ can be written

$$X(e^{j\omega}) = \frac{1}{2}(X(e^{j\omega}) + X^*(e^{-j\omega})) + \frac{1}{2}(X(e^{j\omega}) - X^*(e^{-j\omega}))$$

- The first part is conjugate-symmetric, since

$$\begin{aligned} X_s(e^{j\omega}) &= \frac{1}{2}(X(e^{j\omega}) + X^*(e^{-j\omega})) \\ \Rightarrow \; X_s^*(e^{-j\omega}) &= \frac{1}{2}(X(e^{-j\omega}) + X^*(e^{j\omega}))^* \\ &= \frac{1}{2}(X(e^{j\omega}) + X^*(e^{-j\omega})) = X_s(e^{j\omega}). \end{aligned}$$

- The second part is conjugate-antisymmetric, so

$$X(e^{j\omega}) = X_s(e^{j\omega}) + X_a(e^{j\omega})$$

Symmetry Properties: General $x[n]$

- Suppose $x[n] \leftrightarrow X(e^{j\omega})$.

$$x^*[n] \leftrightarrow X^*(e^{-j\omega})$$

$$x^*[-n] \leftrightarrow X^*(e^{j\omega})$$

$$\text{Re}\{x[n]\} \leftrightarrow X_s(e^{j\omega})$$

$$j\text{Im}\{x[n]\} \leftrightarrow X_a(e^{j\omega})$$

- exemplar proof

$$y[n] = x^*[n] \Rightarrow Y(e^{j\omega}) = \sum_n x^*[n]e^{-j\omega n} = \sum_n (x[n]e^{-j(-\omega)n})^* = X^*(e^{-j\omega})$$

$$z[n] = x^*[-n] \Rightarrow Z(e^{j\omega}) = \sum_n x^*[-n]e^{-j\omega n} = \sum_n (x[-n]e^{-j\omega(-n)})^* = X^*(e^{j\omega})$$

$$w[n] = Re\{x[n]\} = \frac{1}{2}(x[n] + x^*[n]) \Rightarrow W(e^{j\omega}) = \frac{1}{2}(X(e^{j\omega}) + X^*(e^{-j\omega})) = X_s(e^{j\omega})$$

Symmetry Properties: Real $x[n]$

- We have seen that the FT of the real part of $x[n]$ is the conjugate-symmetric part of $X(e^{j\omega})$, and the FT of the imaginary part of $x[n]$ is the conjugate-antisymmetric part of $X(e^{j\omega})$.
- Suppose $x[n]$ is real.

$$
\begin{aligned}
X(e^{j\omega}) &= X^*(e^{-j\omega}) \\
\Rightarrow \quad X_R(e^{j\omega}) &= X_R(e^{-j\omega}) \\
X_I(e^{j\omega}) &= -X_I(e^{-j\omega}) \\
|X(e^{j\omega})| &= |X(e^{-j\omega})| \\
< X(e^{j\omega}) &= - < X(e^{-j\omega})
\end{aligned}
$$

Example

- Let $x[n] = a^n u[n], |a| < 1$. The Fourier transform is

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \frac{1}{1 - ae^{-j\omega}}$$

$$\Rightarrow \quad X(e^{j\omega}) = X^*(e^{-j\omega})$$

$$X_R(e^{j\omega}) = X_R^*(e^{-j\omega}) = \frac{1 - a\cos\omega}{1 + a^2 - 2a\cos\omega}$$

$$X_I(e^{j\omega}) = -X_I(e^{-j\omega}) = \frac{-a\sin\omega}{1 + a^2 - 2a\cos\omega}$$

$$|X(e^{j\omega})| = |X(e^{-j\omega})|$$

$$< X(e^{j\omega}) = - < X(e^{-j\omega}) = \tan^{-1}\left(\frac{-a\sin\omega}{1 - a\cos\omega}\right)$$

Fourier Transform Theorems

- linearity

$$ax[n] + by[n] \leftrightarrow aX(e^{j\omega}) + bY(e^{j\omega})$$

- time shift

$$x[n - n_d] \leftrightarrow e^{-j\omega n_d} X(e^{j\omega})$$

- frequency modulation

$$e^{j\omega_0 n} x[n] \leftrightarrow X(e^{j(\omega - \omega_0)})$$

- time reversal

$$x[-n] \leftrightarrow X(e^{-j\omega})$$

Fourier Transform Theorems

- frequency differentiation

$$nx[n] \leftrightarrow j\frac{dX(e^{j\omega})}{d\omega}$$

- convolution

$$x[n] * h[n] \leftrightarrow X(e^{j\omega})H(e^{j\omega})$$

- multiplication (windowing)

$$x[n]w[n] \leftrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})W(e^{j(\omega-\theta)})d\theta.$$

The integral is a frequency-domain convolution.

Parseval's Theorem

- The convolution of $x[n]$ and $y[n] = x^*[-n]$ has a FT of

$$X(e^{j\omega})Y(e^{j\omega}) = X(e^{j\omega})X^*(e^{j\omega}).$$

- Evaluating $z[n] = x[n] * y[n]$ at $n = 0$, we have

$$z[0] = \sum_k x[k]y[0-k] = \sum_k x[k]x^*[k] = \sum_k |x[k]|^2.$$

- Representing $z[n]$ by the synthesis formula, at evaluating at $n = 0$, we have

$$z[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})X^*(e^{j\omega})e^{j\omega 0}d\omega$$

Energy and Spectrum

- The **energy** of a sequence is the sum of squares of each term. The previous slide shows that

$$E = \sum_{n} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega \ (= z[0]).$$

- We are essentially decompose energy in two ways: time-wise and frequency-wise.
- For this reason, $|X(e^{j\omega})|^2$ is called **energy density spectrum**.

Spectral Relationship

- Suppose we have an LTI system, with frequency response $H(e^{j\omega})$.
- Suppose we have an input $x[n]$, with output $y[n]$.
- Now, from the synthesis formula, we have

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$\Rightarrow y[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})H(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\omega})e^{j\omega}d\omega.$$

It follows that

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}).$$