線性非時變系統

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線性非時變系統 linear time-invariant system LTI

T 為離散時間系統

$$x[n] \longrightarrow \boxed{T} \longrightarrow y[n]$$

如果

$$T\{a_1x_1[n] + a_2x_2[n]\} = a_1y_1[n] + a_2y_2[n],$$

T 稱為線性。如果

$$T\{x[n-n_d]\} = y[n-n_d].$$

稱為非時變。

LTI 回顧

1. 輸出訊號爲輸入訊號與脈衝響應函數之摺積。
2. 因果性系統滿足

$$h[n] = 0, \quad \text{for } n < 0.$$

3. 穩定系統滿足

$$\sum_n |h[n]| < \infty.$$

LTI 系統表示法 representations of LTI systems

1. 時域爲脈衝響應函數 $h[n]$

$$y[n] = h[n] * x[n].$$

2. 頻域爲頻率響應函數 $H(e^{j\omega})$

$$Y(e^{j\omega}) = H(e^{j\omega})\ X(e^{j\omega}).$$

3. z 領域爲系統函數 $H(z)$

$$Y(z) = H(z)\ X(z).$$

頻率響應

在頻域，輸入與輸出訊號的關係爲

$$Y(e^{j\omega}) = H(e^{j\omega})\ X(e^{j\omega}).$$

其振幅與相位關係分別爲

$$|Y(e^{j\omega})| = |H(e^{j\omega})|\ |X(e^{j\omega})|,$$

與

$$\angle Y(e^{j\omega}) = \angle H(e^{j\omega}) + \angle X(e^{j\omega}).$$

- $|H(e^{j\omega})|$ 稱爲振幅響應或增益。
- $\angle H(e^{j\omega})$ 稱爲相位響應或相位移。

Example (理想時延 ideal delay)

理想時延系統在時域爲

$$h_{\text{id}}[n] \triangleq \delta[n-n_d].$$

頻率響應爲

$$H_{\text{id}}(e^{j\omega}) \triangleq \sum_n h_{\text{id}}[n]e^{-j\omega n} = e^{-j\omega n_d}, \quad |\omega| \leq \pi.$$

故振幅響應爲常數

$$|H_{\text{id}}(e^{j\omega})| = 1,$$

相位響應爲線性

$$\angle H_{\text{id}}(e^{j\omega}) = -\omega n_d, \quad |\omega| \leq \pi.$$

Example (理想低通濾波器 ideal low pass filter)

理想低通濾波器在頻域爲

$$H_{\mathrm{lp}}(e^{j\omega}) \triangleq \begin{cases} 1, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \leq \pi, \end{cases}$$

其中 ω_c 爲截止頻率 (cutoff frequency)。脈衝響應函數爲

$$\begin{aligned} h_{\mathrm{lp}}[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{\mathrm{lp}}(e^{j\omega}) e^{j\omega n} d\omega \\ &= \frac{\sin \omega_c n}{\pi n}. \end{aligned}$$

Example (理想高通濾波器 ideal highpass filter)

理想高通濾波器在頻域爲

$$H_{\mathrm{hp}}(e^{j\omega}) \triangleq \begin{cases} 0, & |\omega| < \omega_c, \\ 1, & \omega_c < |\omega| \leq \pi. \end{cases}$$

利用理想低通濾波器，則

$$H_{\mathrm{hp}}(e^{j\omega}) = 1 - H_{\mathrm{lp}}(e^{j\omega})$$

$$\xrightarrow{\text{IDTFT}} \quad h_{\mathrm{hp}}[n] = \delta[n] - h_{\mathrm{lp}}[n] = \delta[n] - \frac{\sin \omega_c n}{\pi n}.$$

波群延遲 group delay

LTI 系統的波群延遲定義為

$$\operatorname{grd} H(e^{j\omega}) \triangleq -\frac{d}{d\omega} \arg H(e^{j\omega}),$$

其中 $\arg H(e^{j\omega})$ 為連續相位 (continuous phase)。連續相位與相位 $\angle H(e^{j\omega})$ 的關係為

$$\arg H(e^{j\omega}) = \angle H(e^{j\omega}) + 2\pi k, \quad k \in \mathcal{Z}.$$

物理意義

$s[n]\cos(\omega_0 n)$ 爲窄頻帶訊號。在 ω_0 附近，

$$\begin{aligned}\arg H(e^{j\omega}) &\approx -\phi_0 + \omega \frac{d}{d\omega} \arg H(e^{j\omega})|_{\omega_0} \\ &= -\phi_0 - \omega \operatorname{grd} H(e^{j\omega})|_{\omega_0} \\ &= -\phi_0 - \omega n_d, \quad \text{where } n_d \triangleq \operatorname{grd} H(e^{j\omega})|_{\omega_0}.\end{aligned}$$

忽略 ϕ_0，可得

$$\xrightarrow{s[n]\cos(\omega_0 n)} \boxed{|H(e^{j\omega_0})|} \xrightarrow{|H(e^{j\omega_0})|s[n]\cos(\omega_0 n)}$$

$$\xrightarrow{|H(e^{j\omega_0})|s[n]\cos(\omega_0 n)} \boxed{\delta[n-n_d]} \xrightarrow{|H(e^{j\omega_0})|s[n-n_d]\cos(\omega_0(n-n_d))}$$

也就是波群包絡線 $s[n]$ 延遲了 n_d。

有理化系統函數 rational system function

線性常係數差分方程

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k].$$

作 z 轉換，可得 $\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$，故其系統函數爲

$$H(z) \triangleq \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

稱爲**有理化系統函數**。

極點與零點 pole and zero

有理化系統函數也可以表示成

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})},$$

其中

1. $(1 - c_k z^{-1})$ 貢獻零點於 $z = c_k$，極點於 $z = 0$
2. $(1 - d_k z^{-1})$ 貢獻極點於 $z = d_k$，零點於 $z = 0$

反向問題

給定 $H(z)$，決定對應的 LTI 系統。

收斂區域的選擇

光是 $H(z)$ 無法決定系統，還必須指定收斂區域。

1. 穩定系統，則收斂區域必須包括單位圓。
2. 因果性系統，收斂區域必須向外延伸至 ∞。

Example

考慮 LCCDE

$$y[n] - \frac{5}{2}y[n-1] + y[n-2] = x[n].$$

系統函數爲

$$H(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}.$$

極點爲

$$d_1 = \frac{1}{2}, \quad d_2 = 2.$$

指定收斂區域爲 $|z| > 2$、$2 > |z| > \frac{1}{2}$、或是 $|z| < \frac{1}{2}$ 將導致不同 LTI 系統。

逆系統 inverse systems

逆系統之系統函數爲

$$H_i(z) = \frac{1}{H(z)}.$$

1. 時域方程式爲

$$h_i[n] * h[n] = h[n] * h_i[n] = \delta[n].$$

2. 兩個系統的收斂區域必須有交集。

有理系統函數系統之逆系統

有理系統函數

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}.$$

逆系統滿足

$$H_i(z) = \frac{1}{H(z)} = \left(\frac{a_0}{b_0}\right) \frac{\prod_{k=1}^{N}(1 - d_k z^{-1})}{\prod_{k=1}^{M}(1 - c_k z^{-1})}.$$

$H(z)$ 與 $H_i(z)$ 交換其極點與零點。

Example

因果性系統

$$H(z) = \frac{1 - 0.5z^{-1}}{1 - 0.9z^{-1}}, \quad \text{ROC: } |z| > 0.9.$$

逆系統之系統函數爲

$$H_i(z) = \frac{1 - 0.9z^{-1}}{1 - 0.5z^{-1}}.$$

爲了與 $H(z)$ 的收斂區域有交集，$H_i(z)$ 的收斂區域必須爲 $|z| > 0.5$。

Example

$$H(z) = \frac{z^{-1} - 0.5}{1 - 0.9z^{-1}}, \quad \text{ROC: } |z| > 0.9.$$

逆系統之系統函數爲

$$H_i(z) = \frac{1 - 0.9z^{-1}}{z^{-1} - 0.5} = \frac{-2 + 1.8z^{-1}}{1 - 2z^{-1}}.$$

$|z| > 2$ 與 $|z| < 2$ 均與 $|z| > 0.9$ 有交集。故存在兩個逆系統。

部分分式展開 partial fraction expansion

僅有一階極點的有理系統函數

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})},$$

可以寫成

$$H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}.$$

無限脈衝響應 infinite impulse response

若存在 $A_k \neq 0$，$h[n]$ 非有限長度，稱爲無限脈衝響應系統。

Example

常係數差分方程

$$y[n] - ay[n-1] = x[n]$$

對應的系統函數爲

$$H(z) = \frac{1}{1 - az^{-1}}.$$

若爲因果性系統，則

$$h[n] = a^n u[n].$$

有限脈衝響應 finite impulse response FIR

$N = 0$ 時系統函數爲

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}.$$

脈衝響應函數爲

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k] = \begin{cases} b_n, & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases}.$$

稱爲有限脈衝響應系統。

Example

系統函數

$$H(z) = \sum_{k=0}^{M} a^k z^{-k}$$

對應的脈衝響應函數爲

$$h[n] = \sum_{k=0}^{M} a^k \delta[n-k].$$

此系統可以 LCCDE 描述

$$y[n] = \sum_{k=0}^{M} a^k x[n-k].$$

頻率響應 frequency response

有理系統函數之穩定 LTI 系統，頻率響應爲

$$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}} = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}.$$

以零點與極點表示，則是

$$H(e^{j\omega}) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k e^{-j\omega})}{\prod_{k=1}^{N}(1 - d_k e^{-j\omega})}.$$

增益 gain

$$|H(e^{j\omega})| = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=1}^{M} |1 - c_k e^{-j\omega}|}{\prod_{k=1}^{N} |1 - d_k e^{-j\omega}|}$$

分貝 decibel

增益可以分貝表示 (dB)

$$\text{gain in dB} \triangleq 10 \log_{10} |H(e^{j\omega})|^2 = 20 \log_{10} |H(e^{j\omega})|.$$

上述系統增益的 dB 值爲

$$20 \log_{10} \left|\frac{b_0}{a_0}\right| + \sum_{k=1}^{M} 20 \log_{10} |1 - c_k e^{-j\omega}| - \sum_{k=1}^{N} 20 \log_{10} |1 - d_k e^{-j\omega}|.$$

相位 phase

$$\angle H(e^{j\omega}) = \angle\left(\frac{b_0}{a_0}\right) + \sum_{k=1}^{M} \angle(1 - c_k e^{-j\omega}) - \sum_{k=1}^{N} \angle(1 - d_k e^{-j\omega}).$$

波群延遲 group delay

$$\begin{aligned}\operatorname{grd} H(e^{j\omega}) &\triangleq -\frac{d}{d\omega} \arg H(e^{j\omega}) \\ &= \sum_{k=1}^{N} \frac{d}{d\omega} \arg(1 - d_k e^{-j\omega}) - \sum_{k=1}^{M} \frac{d}{d\omega} \arg(1 - c_k e^{-j\omega}).\end{aligned}$$

note

極點與零點對於增益、相位、波群延遲的貢獻類似，符號相反。

極點對於波群延遲的貢獻 contribution by a pole to the group delay

極點為 $d = re^{j\theta}$。

$$\left\{\arg[1 - re^{j(\theta-\omega)}]\right\}' = \left[\tan^{-1}\frac{-r\sin(\theta-\omega)}{1-r\cos(\theta-\omega)}\right]'$$

$$= \frac{1}{1+\left[\frac{-r\sin(\theta-\omega)}{1-r\cos(\theta-\omega)}\right]^2}\left[\frac{-r\sin(\theta-\omega)}{1-r\cos(\theta-\omega)}\right]'$$

$$= \frac{[-r\sin(\theta-\omega)]'[1-r\cos(\theta-\omega)] - [-r\sin(\theta-\omega)][1-r\cos(\theta-\omega)]'}{\left(1+\left[\frac{-r\sin(\theta-\omega)}{1-r\cos(\theta-\omega)}\right]^2\right)[1-r\cos(\theta-\omega)]^2}$$

$$= \frac{[r\cos(\theta-\omega)][1-r\cos(\theta-\omega)] - [-r\sin(\theta-\omega)][-r\sin(\theta-\omega)]}{[1-r\cos(\theta-\omega)]^2 + [-r\sin(\theta-\omega)]^2}$$

$$= \frac{-r^2 + r\cos(\theta-\omega)}{1+r^2-2r\cos(\theta-\omega)} = \frac{-|d|^2 + Re\{de^{-j\omega}\}}{1+|d|^2 - 2\ Re\{de^{-j\omega}\}}.$$

零點對於增益的貢獻 contribution by a zero to the gain

考慮有理系統函數分子的一項 $(1 - cz^{-1})$，其中 $c = re^{j\theta}$。增益的平方爲

$$|1 - ce^{-j\omega}|^2 = |1 - re^{j\theta}e^{-j\omega}|^2 = (1 - re^{j\theta}e^{-j\omega})(1 - re^{-j\theta}e^{j\omega})$$
$$= 1 + r^2 - 2r\cos(\omega - \theta).$$

顯然增益在 $\omega = \theta$ 爲最低，在 $\omega = \theta \pm \pi$ 爲最高。

Example

當 $r = 0.9$，最大增益之 dB 值爲

$$20\log_{10}(1 + 0.9) = 5.57,$$

最小增益之 dB 值爲

$$20\log_{10}(1 - 0.9) = -20.$$

零點對於相位的貢獻 contribution by a zero to the phase

$$\angle\left[1 - re^{j\theta}e^{-j\omega}\right] = \tan^{-1}\left[\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right].$$

零點對於波群延遲的貢獻 contribution by a zero to the group delay

$$\operatorname{grd}\left[1 - re^{j\theta}e^{-j\omega}\right] = \frac{r^2 - r\cos(\omega-\theta)}{1+r^2-2r\cos(\omega-\theta)}.$$

系統設計 system design

給定振幅響應 $|H(e^{j\omega})|$，相位通常不計，設計系統 $H(z)$。

輔助函數 auxiliary function

振幅響應之平方函數爲

$$|H(e^{j\omega})|^2 = H(e^{j\omega})H^*(e^{j\omega}) = H(z)H^*\left(\frac{1}{z^*}\right)\bigg|_{z=e^{j\omega}} = C(z)|_{z=e^{j\omega}}.$$

其中 $C(z) \triangleq H(z)\, H^*\left(\frac{1}{z^*}\right)$ 稱爲**輔助函數**。

有理系統函數設計

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}.$$

取 z 的共軛、倒數，再將 $H(z)$ 取共軛，則

$$H^*\left(\frac{1}{z^*}\right) = \left(\frac{b_0}{a_0}\right) \left(\frac{\prod_{k=1}^{M}(1 - c_k z^*)}{\prod_{k=1}^{N}(1 - d_k z^*)} \right)^* = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k^* z)}{\prod_{k=1}^{N}(1 - d_k^* z)}.$$

所以輔助函數爲

$$C(z) = H(z)\, H^*\left(\frac{1}{z^*}\right) = \left(\frac{b_0}{a_0}\right)^2 \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})(1 - c_k^* z)}{\prod_{k=1}^{N}(1 - d_k z^{-1})(1 - d_k^* z)}.$$

$C(z)$ 的極點成共軛倒數對 $(d_k, (d_k^*)^{-1})$。

$$d_k = re^{j\theta} \xrightarrow{\text{conjugation}} d_k^* = re^{-j\theta} \xrightarrow{\text{reciprocal}} (d_k^*)^{-1} = \frac{1}{r} e^{j\theta}.$$

$C(z)$ 的零點亦成共軛倒數對 $(c_k, (c_k^*)^{-1})$。

最小相位系統附加條件 minimum-phase systems

給定 $|H(e^{j\omega})|$ 之後，爲了決定 $H(e^{j\omega})$，需要加上一些條件。**最小相位系統**的整個決定過程如下

1. 以 z 取代 $|H(e^{j\omega})|^2$ 中的 $e^{j\omega}$，得到 $C(z)$。
2. 將 $C(z)$ 極點與零點湊成對，一個屬於 $H(z)$，另一個屬於 $H^*\left(\frac{1}{z^*}\right)$。
3. 令系統爲穩定性與因果性。因此，單位圓內的極點屬於 $H(z)$。
4. 令逆系統爲穩定性與因果性。因此，單位圓內的零點屬於 $H(z)$。

Example (5.12)

Given $C(z)$ (from $|H(e^{j\omega})|$), we want to decide $H(z)$.

- The poles are in conjugate reciprocal pairs (P_1, P_4), (P_2, P_5), and (P_3, P_6). The zeros are in conjugate reciprocal pairs (Z_1, Z_4), (Z_2, Z_5), and (Z_3, Z_6).
- If the system is causal and stable, the poles of $H(z)$ must be P_1, P_2, P_3.
- The zeros can be similarly decided.

一階全通系統 first-order all-pass systems

一階全通系統系統函數爲

$$H_{\text{ap}}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}.$$

1. 極點與零點爲共軛倒數對。
2. 穩定性與因果性一階全通系統

$$|a| < 1.$$

頻率響應爲

$$H_{\text{ap}}(e^{j\omega}) \triangleq H_{\text{ap}}(z)\big|_{z=e^{j\omega}} = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}}.$$

1. 振幅響應爲

$$\left|H_{\mathrm{ap}}(e^{j\omega})\right| = \left|\frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}}\right| = \left|e^{-j\omega}\right| \left|\frac{1 - a^* e^{j\omega}}{1 - ae^{-j\omega}}\right| = 1.$$

2. 相位響應爲

$$\begin{aligned}\angle\left(e^{-j\omega}\frac{1 - re^{j(\omega-\theta)}}{1 - re^{-j(\omega-\theta)}}\right) &= \angle\, e^{-j\omega} + \angle(1 - re^{j(\omega-\theta)}) - \angle(1 - re^{-j(\omega-\theta)}) \\ &= -\omega - 2\tan^{-1}\left[\frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)}\right].\end{aligned}$$

3. 波群延遲爲

$$\mathrm{grd}\left[\frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}\right] = \frac{1 - r^2}{|1 - re^{j\theta}e^{-j\omega}|^2}.$$

穩定性與因果性一階全通系統波群延遲大於 0。

高階全通系統

實脈衝響應函數之高階全通系統系統函數爲

$$H_{ap}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}.$$

實數對 (d_k, d_k^{-1}) 爲極點與零點、共軛複數對 (e_k, e_k^*) 爲極點、$((e_k^*)^{-1}, (e_k)^{-1})$ 爲零點。

全通系統之相位

穩定性與因果性全通系統之連續相位 $\arg H_{\text{ap}}(e^{j\omega})$ 在 $0 < \omega < \pi$ 區間爲非負。

證明

全通系統在 $\omega = 0$ 的相位爲

$$\arg H_{\text{ap}}(e^{j0}) = \arg \left(A \prod_{k=1}^{M_r} \frac{1 - d_k}{1 - d_k} \prod_{k=1}^{M_c} \frac{(1 - e_k^*)(1 - e_k)}{(1 - e_k)(1 - e_k^*)} \right) = 0.$$

The group delay is the sum of the group delays of first-order all-pass systems, which are positive if the all-pass system is causal and stable. As a result, the slope of the continuous phase is negative, meaning that $\arg H_{\text{ap}}(e^{j\omega})$ cannot be positive.

零點的反射 reflection of a zero

Suppose all poles and zeros of $H(z)$ are inside the unit circle except for one zero, say at $z_0 = \frac{1}{c^*}$, $|c| < 1$. $H(z)$ can be expressed as follows:

$$\begin{aligned} H(z) &= H_1(z)(z^{-1} - c^*) \\ &= H_1(z)(z^{-1} - c^*)\left(\frac{1 - cz^{-1}}{1 - cz^{-1}}\right) \\ &= \left[H_1(z)(1 - cz^{-1})\right]\left(\frac{z^{-1} - c^*}{1 - cz^{-1}}\right) \end{aligned}$$

The difference between $H(z) = H_1(z)(z^{-1} - c^*)$ and $H_1(z)(1 - cz^{-1})$ is that a zero of $H(z)$ at $z_0 = \frac{1}{c^*}$, $|c| < 1$ outside the unit circle is moved to a zero of $H_1(z)(1 - cz^{-1})$ at $z_0' = c$ inside the unit circle.

最小相位性質

Note that $H_1(z)(1 - cz^{-1})$ is minimum-phase since

- all poles and zeros of $H_1(z)$ are inside the unit circle, and
- there is one more zero at $z_0' = c$ for $H_1(z)(1 - cz^{-1})$.

系統函數之分解

1. By reflecting the zeros outside the unit circle one by one, we move all zeros inside the unit circle.
2. The entire sequence of moving the zeros is equivalent to a causal and stable all-pass system $H_{\mathrm{ap}}(z)$.

Thus, for any causal and stable system with rational system function, the system function can be expressed as

$$H(z) = H_{\min}(z) H_{\mathrm{ap}}(z).$$

Example

Consider the rational system function

$$H(z) = \frac{1 + 3z^{-1}}{1 + \frac{1}{2}z^{-1}}.$$

We want to move the zero $z_0 = -3$ inside the unit circle, with

$$z_0 = (c^*)^{-1} = -3 \quad \Rightarrow \quad z_0' = c = -\frac{1}{3}.$$

Introducing the all-pass system $H_{ap}(z) = \frac{z^{-1} - c^*}{1 - cz^{-1}} = \frac{z^{-1} + \frac{1}{3}}{1 + \frac{1}{3}z^{-1}}$, we have

$$H(z) = 3\left(\frac{1 + \frac{1}{3}z^{-1}}{1 + \frac{1}{2}z^{-1}}\right)\left(\frac{z^{-1} + \frac{1}{3}}{1 + \frac{1}{3}z^{-1}}\right) = H_{min}(z)H_{ap}(z).$$

Example (second-order case)

Consider the system function

$$H(z) = \frac{(1 + \frac{3}{2}e^{j\pi/4}z^{-1})(1 + \frac{3}{2}e^{-j\pi/4}z^{-1})}{1 - \frac{1}{3}z^{-1}}.$$

The zeros outside the unit circle are

$$z_{0,\pm} = \frac{1}{c_{\pm}^*} = -\frac{3}{2}e^{\pm j\pi/4}.$$

We want to move them inside the unit circle to

$$z'_{0,\pm} = c_{\pm} = -\frac{2}{3}e^{\pm j\pi/4}.$$

Example (continue)

Introducing the all-pass system

$$H_{\text{ap}}(z) = \frac{z^{-1} - c_+^*}{1 - c_+ z^{-1}} \cdot \frac{z^{-1} - c_-^*}{1 - c_- z^{-1}}$$
$$= \frac{(z^{-1} + \frac{2}{3}e^{-j\pi/4})(z^{-1} + \frac{2}{3}e^{j\pi/4})}{(1 + \frac{2}{3}e^{j\pi/4}z^{-1})(1 + \frac{2}{3}e^{-j\pi/4}z^{-1})},$$

we have

$$H(z) = \frac{9}{4}\left[\frac{(1 + \frac{2}{3}e^{j\pi/4}z^{-1})(1 + \frac{2}{3}e^{-j\pi/4}z^{-1})}{1 - \frac{1}{3}z^{-1}}\right]$$
$$\times \left[\frac{(z^{-1} + \frac{2}{3}e^{-j\pi/4})(z^{-1} + \frac{2}{3}e^{j\pi/4})}{(1 + \frac{2}{3}e^{j\pi/4}z^{-1})(1 + \frac{2}{3}e^{-j\pi/4}z^{-1})}\right]$$
$$= H_{\min}(z)H_{\text{ap}}(z).$$

最大相位與最小波群延遲性 maximum-phase and minimum group-delay properties

The minimum-phase system $H_{\text{min}}(z)$ has the maximum phase and the minimum group delay among all causal and stable systems with the same magnitude response.

證明

A causal and stable system with rational function can be expressed as

$$H(z) = H_{\text{min}}(z) H_{\text{ap}}(z).$$

The corresponding phase and group-delay relations are

$$\begin{aligned}\arg H(e^{j\omega}) &= \arg H_{\text{min}}(e^{j\omega}) + \arg H_{\text{ap}}(e^{j\omega}),\\ \text{grd}[H(e^{j\omega})] &= \text{grd}[H_{\text{min}}(e^{j\omega})] + \text{grd}[H_{\text{ap}}(e^{j\omega})].\end{aligned}$$

A stable and causal all-pass system has a positive group delay and a negative phase. The properties follow.

線性相位系統 linear phase system

Consider an LTI system with the following frequency response

$$H(e^{j\omega}) = e^{-j\omega\alpha}, \quad |\omega| < \pi.$$

This system has a linear phase, and constant group delay

$$|H(e^{j\omega})| = 1, \quad \angle H(e^{j\omega}) = -\omega\alpha, \quad \text{grd}[H(e^{j\omega})] = \alpha.$$

It corresponds to an ideal delay

$$y[n] = x_c(nT - \alpha T).$$

一般化線性相位 generalized linear phase

一般化線性相位頻率響應爲以下形式

$$H(e^{j\omega}) = A(e^{j\omega})\ e^{-j\omega\alpha + j\beta},$$

其中 α, β 爲實數，$A(e^{j\omega})$ 爲實函數。其實部與虛部分別爲

$$\begin{aligned} H(e^{j\omega}) &= A(e^{j\omega})\ e^{-j\omega\alpha + j\beta} \\ &= A(e^{j\omega}) \cos(\beta - \omega\alpha) + jA(e^{j\omega}) \sin(\beta - \omega\alpha). \end{aligned}$$

性質

$$H(e^{j\omega}) = \sum_n h[n]e^{-j\omega n} = \sum_n h[n]\cos\omega n - j\sum_n h[n]\sin\omega n.$$

利用前式，

$$\frac{\sin(\beta-\omega\alpha)}{\cos(\beta-\omega\alpha)} = -\frac{\sum_n h[n]\sin\omega n}{\sum_n h[n]\cos\omega n}.$$

因此

$$\sum_n h[n][\cos(\beta-\omega\alpha)\sin\omega n + \sin(\beta-\omega\alpha)\cos\omega n] = 0,$$

相當於

$$\sum_n h[n]\sin[\omega(n-\alpha)+\beta] = 0, \quad \forall\ \omega.$$

線性相位因果性有限脈衝響應函數系統

因果性有限脈衝響應函數系統可以表示成

$$h[n] = \begin{cases} h_n, & 0 \leq n \leq M, \\ 0, & \text{otherwise.} \end{cases}$$

以下介紹 4 種線性相位系統。

第 1 類線性相位系統 Type I linear-phase system

第 1 類線性相位系統有對稱的脈衝響應函數

$$h_n = h_{M-n}, \quad 0 \leq n \leq M,$$

且 M 爲偶數。

第 2 類線性相位系統 Type II linear-phase system

第 2 類線性相位系統亦有對稱的脈衝響應函數

$$h_n = h_{M-n}, \quad 0 \leq n \leq M,$$

但 M 爲奇數。

Type I. Let $M = 2\alpha$.

$$
\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M} h[n] e^{-j\omega n} \\
&= \left(\sum_{n=0}^{\alpha-1} + \sum_{n=\alpha+1}^{M} \right) h[n] e^{-j\omega n} + \overbrace{h[\alpha] e^{-j\omega\alpha}}^{\text{middle term}} \\
&= \sum_{k=1}^{\alpha} \left(h[\alpha-k] e^{-j\omega(\alpha-k)} + h[\alpha+k] e^{-j\omega(\alpha+k)} \right) + h[\alpha] e^{-j\omega\alpha} \\
&= \sum_{k=1}^{\alpha} h_{\alpha-k} \left(e^{-j\omega(\alpha-k)} + e^{-j\omega(\alpha+k)} \right) + h_\alpha e^{-j\omega\alpha} \\
&= \sum_{k=1}^{\alpha} h_{\alpha-k} e^{-j\omega\alpha} \left(e^{j\omega k} + e^{-j\omega k} \right) + h_\alpha e^{-j\omega\alpha} \\
&= \left(\sum_{k=0}^{\alpha} a[k] \cos \omega k \right) e^{-j\omega\alpha}, \quad a[0] \triangleq h_\alpha, \quad a[k] \triangleq 2h_{\alpha-k}.
\end{aligned}
$$

Type II. Let $M = 2\alpha$.

$$
\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M=2\alpha} h[n] e^{-j\omega n} \\
&= \left(\sum_{n=0}^{\alpha-1/2} + \sum_{n=\alpha+1/2}^{M} \right) h[n] e^{-j\omega n} \\
&= \sum_{k=1}^{\alpha+1/2} \left(h\left[\alpha + \frac{1}{2} - k\right] e^{-j\omega(\alpha+\frac{1}{2}-k)} + h\left[\alpha - \frac{1}{2} + k\right] e^{-j\omega(\alpha-\frac{1}{2}+k)} \right) \\
&= \sum_{k=1}^{\alpha+1/2} h_{\alpha+\frac{1}{2}-k} \left(e^{-j\omega(\alpha+\frac{1}{2}-k)} + e^{-j\omega(\alpha-\frac{1}{2}+k)} \right) \\
&= \sum_{k=1}^{\alpha+1/2} h_{\alpha+\frac{1}{2}-k} e^{-j\omega\alpha} \left(e^{j\omega(k-\frac{1}{2})} + e^{-j\omega(k-\frac{1}{2})} \right) \\
&= e^{-j\omega\alpha} \sum_{k=1}^{\alpha+1/2} 2h_{\alpha+\frac{1}{2}-k} \cos\left[\omega\left(k - \frac{1}{2}\right)\right] \\
&= \left(\sum_{k=1}^{\alpha+1/2} b[k] \cos\left[\omega\left(k - \frac{1}{2}\right)\right] \right) e^{-j\omega\alpha}, \quad b[k] \triangleq 2h_{\alpha+\frac{1}{2}-k}.
\end{aligned}
$$

第 3 類線性相位系統 Type III linear-phase system

第 3 類線性相位系統有反對稱的脈衝響應函數

$$h_n = -h_{M-n}, \quad 0 \leq n \leq M,$$

且 M 為偶數。

第 4 類線性相位系統 Type IV linear-phase system

第 4 類線性相位系統亦有反對稱的脈衝響應函數

$$h_n = -h_{M-n}, \quad 0 \leq n \leq M,$$

但 M 為奇數。

Type III. Let $M = 2\alpha$.

$$
\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M=2\alpha} h[n]e^{-j\omega n} \\
&= \left(\sum_{n=0}^{\alpha-1} + \sum_{n=\alpha+1}^{M} \right) h[n]e^{-j\omega n} \\
&= \sum_{k=1}^{\alpha} \left(h[\alpha-k]e^{-j\omega(\alpha-k)} + h[\alpha+k]e^{-j\omega(\alpha+k)} \right) \\
&= \sum_{k=1}^{\alpha} h_{\alpha-k} \left(e^{-j\omega(\alpha-k)} - e^{-j\omega(\alpha+k)} \right) \\
&= \sum_{k=1}^{\alpha} h_{\alpha-k} e^{-j\omega\alpha} \left(e^{j\omega k} - e^{-j\omega k} \right) \\
&= je^{-j\omega\alpha} \left(\sum_{k=1}^{\alpha} c[k] \sin \omega k \right) \\
&= j \left(\sum_{k=1}^{\alpha} c[k] \sin \omega k \right) e^{-j\omega\alpha}, \quad c[k] \triangleq 2h_{\alpha-k}.
\end{aligned}
$$

Type IV. Let $M = 2\alpha$.

$$
\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M=2\alpha} h[n] e^{-j\omega n} \\
&= \left(\sum_{n=0}^{\alpha-1/2} + \sum_{n=\alpha+1/2}^{M} \right) h[n] e^{-j\omega n} \\
&= \sum_{k=1}^{\alpha+1/2} \left(h\left[\alpha + \frac{1}{2} - k\right] e^{-j\omega(\alpha+\frac{1}{2}-k)} + h\left[\alpha - \frac{1}{2} + k\right] e^{-j\omega(\alpha-\frac{1}{2}+k)} \right) \\
&= \sum_{k=1}^{\alpha+1/2} h_{\alpha+\frac{1}{2}-k} \left(e^{-j\omega(\alpha+\frac{1}{2}-k)} - e^{-j\omega(\alpha-\frac{1}{2}+k)} \right) \\
&= \sum_{k=1}^{\alpha+1/2} h_{\alpha+\frac{1}{2}-k} e^{-j\omega\alpha} \left(e^{j\omega(k-\frac{1}{2})} - e^{-j\omega(k-\frac{1}{2})} \right) \\
&= j e^{-j\omega\alpha} \sum_{k=1}^{\alpha+1/2} 2h_{\alpha+\frac{1}{2}-k} \sin\left[\omega\left(k - \frac{1}{2}\right)\right] \\
&= j \left(\sum_{k=1}^{\alpha+1/2} d[k] \sin\left[\omega\left(k - \frac{1}{2}\right)\right] \right) e^{-j\omega\alpha}, \quad d[k] \triangleq 2h_{\alpha+\frac{1}{2}-k}.
\end{aligned}
$$