1. (20%) Let X and Y be discrete random variables taking value in $\{1, 2, 3\}$. Furthermore, the joint probability satisfies

$$p(x, y) \propto 2^{|x-y|}.$$

Compute $H(X, Y), H(Y), H(X|Y)$ and $D(p(x, y)||p(x)p(y))$.

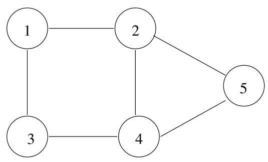
Solution:

$$p = \frac{1}{19}\begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 1 \end{bmatrix}$$

From this one can compute

$$\begin{aligned} H(X, Y) &= \log 19 - \frac{24}{19}, \\ H(Y) &= \log 19 - \frac{14}{19}\log 7 - \frac{5}{19}\log 5, \\ H(X|Y) &= H(X, Y) - H(Y), \\ D(p(x, y)||p(x)p(y)) &= I(X; Y) = H(X) + H(Y) - H(X, Y) \end{aligned}$$

2. (20%) Compute the entropy rate of a random walk on the graph



(a) if the weights of all edges are equal,

(b) if the weight of the edge between node i and node j is $|i - j|$.

Solution:

(a) From (4.41), $\log 12 - H(\frac{2}{12}, \frac{3}{12}, \frac{2}{12}, \frac{3}{12}, \frac{2}{12}) = \frac{1}{2}\log 6$

(b) From (4.40), $(\log 20 - \frac{2}{5} - \frac{3}{10}\log 3) - (\log 20 - \frac{7}{10} - \frac{3}{5}\log 3) = \frac{3}{10}\log 6$

3. (10%) Let $X_1, \ldots, X_n$ be i.i.d. random variables drawn from $p(x)$. Let $q(x_1, \ldots, x_n) = \prod_{i=1}^n q(x_i)$. What is

$$\lim_{n\to\infty} -\frac{1}{n}\log\frac{p(X_1, \ldots, X_n)}{q(X_1, \ldots, X_n)}?$$

Solution:

$$\lim_{n\to\infty} \frac{1}{n}\sum_i \log\frac{p(x_i)}{q(x_i)} \to E\log\frac{p(X)}{q(X)} = D(p||q).$$

4. (10%) Let $H(\mathcal{V})$ be the entropy rate of the stationary stochastic process $\{V_n | n = 1, 2, \dots\}$. Show that

$$H(\mathcal{V}) \leq \frac{H(V_1, \dots, V_k)}{k}.$$

Proof:

$$\frac{H(V_1, \dots, V_k)}{k} = \frac{\sum_{i=1}^{k} H(V_i | V_{1:i-1})}{k} \geq \frac{\sum_{i=1}^{k} H(\mathcal{V})}{k} = H(\mathcal{V}).$$

5. (20%) Consider the discrete memoryless channel $Y = X + Z \pmod{10}$, where $Z \in \{-1, 0, 1\}$ and $p(Z)$ is uniform, and $X \in \{0, \dots, 9\}$.

   (a) What is the capacity? **Ans:** $\log \frac{10}{3}$.

   (b) What is $p^*(x)$ that achieves this capacity? **Ans:** Uniform.

6. (20%) Given $p = (0.49, 0.26, 0.12, 0.04, 0.04, 0.05)$.

   (a) Find a binary prefix code with the minimum expected codeword length. **Solution:** Use Huffman code.

   (b) Find another binary prefix code with a longer expected codeword length. **Solution:** Just add something based on Huffman code.