Final Exam 2022.01.05

1. Solve the following system of linear equations

$$\begin{cases} x - y + z = 0 \\ x + 2y - z = 1 \\ x + y + z = 3 \end{cases}$$

2. Solve the following under-determined system of linear equations

$$\begin{cases} x_1 + 4x_2 + 6x_3 + 3x_4 = 0 \\ x_1 + 5x_2 + 7x_3 + 2x_4 = 1 \\ x_1 + 3x_2 + 5x_3 + x_4 = 3 \end{cases}$$

3. Solve the following over-determined system of linear equations

$$\begin{cases} x_1 + x_2 = 0 \\ x_1 + 2x_2 = 1 \\ x_1 + 3x_2 = 3 \end{cases}$$

4. Derive the projection matrix $\boldsymbol{P}$ to the column space of

$$\boldsymbol{B} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -1 & 1 \end{bmatrix}$$

5. Compute $F_3(2\%), F_{13}(3\%), F_{100}(5\%)$ where F_n is the determinant of $n \times n$ tridiagonal matrix

$$F_n = \begin{vmatrix} 1 & -1 & & & \\ 1 & 1 & -1 & & \\ & 1 & 1 & -1 & \\ & & \cdot & \cdot & \cdot \\ & & & 1 & 1 \end{vmatrix}$$

6. Show that $\boldsymbol{G}^2 - (a+d)\boldsymbol{G} + (ad - bc)\boldsymbol{I} = \boldsymbol{0}$ where

$$\boldsymbol{G} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

7. Derive the diagonalization $\boldsymbol{V} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^H$ where $\boldsymbol{U}$ is unitary, $\boldsymbol{\Lambda}$ is diagonal, and

$$\boldsymbol{V} = \begin{bmatrix} 1 & 1-i \\ 1+i & 1 \end{bmatrix}$$

8. Derive $e^{\boldsymbol{P}}$ where

$$\boldsymbol{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

9. Solve linear differential equation

$$\frac{d\boldsymbol{u}}{dt} = \boldsymbol{A}\boldsymbol{u}, \text{ where } \boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } \boldsymbol{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

10. Derive matrix representation for the linear transformation consisting of the counter-clockwise rotation of $\frac{\pi}{4}$ followed by the projection to the line $y = x$.

    (a) in standard basis

    (b) in orthonormal eigenbasis of the projection