Arrival Processes

Chia-Ping Chen

Professor
Department of Computer Science and Engineering
National Sun Yat-sen University

Probability
A **random process** is a random experiment which evolves in time and generates a set of random values as outcome.

**Sample path.** Every $\omega \in \Omega$ corresponds to a sequence or function $X(\omega)$, which is called a **sample path** or **sample function**.
A random process can be seen as a collection of random variables indexed by time.

**Classification.** The index set may be discrete or continuous.

- A **discrete-time** random process has a discrete index set.
- A **continuous-time** random process has a continuous index set.
Every index of a random process corresponds to a random variable.

**Notation.** Let $X$ be a random process.

- If $X$ is discrete-time, the random variable indexed by time $n$ is denoted by $X_n$.
- If $X$ is continuous-time, the random variable indexed by time $t$ is denoted by $X_t$. 
Bernoulli Processes
An **arrival process** is a collection of Bernoulli random variables, where value 1 is used for arrival and value 0 is used for no arrival.

**Examples.** An arrival process can be used to model

- speech activity
- winning
- renewal
A Bernoulli process is a discrete-time arrival process with a collection of i.i.d. Bernoulli random variables.

The Bernoulli process $X$ with $X_n \sim \text{Bernoulli}(p)$ is denoted by

$$X \sim \text{BerPro}(p)$$
Example 6.1

Let $X$ be a Bernoulli process.

- Random variable $U$ is the number of arrivals from time 1 to 5, and random variable $V$ is the number of arrivals from time 6 to 10. Then $U \perp \perp V$.

- Random variable $W$ is the first odd-index time with an arrival, and random variable $Y$ is the first even-index time with an arrival. Then $W \perp \perp Y$. 
Random variables are associated with a Bernoulli process.

Consider $X \sim \text{BerPro}(p)$.

- The number of arrivals from time 1 to time $n$ is a binomial random variable
  \[ S_n \sim \text{binomial}(n, p) \]

- The first arrival time is a geometric random variable
  \[ T \sim \text{geometric}(p) \]
A re-started Bernoulli process is a Bernoulli process.

Consider $X \sim \text{BerPro}(p)$.

- Suppose $X$ is restarted at time $l \geq 0$. The restarted process $X' : X'_n = X_{l+n}$ is $\text{BerPro}(p)$.

- Suppose $X$ is restarted at a random time $L \geq 0$. If $L$ depends only on $X_1, \ldots, X_L$, then the restarted process $X' : X'_n = X_{L+n}$ is $\text{BerPro}(p)$. 
Example 6.2

A computer executes 2 types of jobs, **priority** and **non-priority**. A slot is **busy** if the computer executes a priority job, **idle** otherwise. We call a string of idle slots, flanked by busy slots, an **idle period**. We call a string of busy slots, flanked by idle slots, a **busy period**.

A priority job occurs with probability $p$ at the beginning of each slot, independent of other slots, and requires one full slot to execute. A non-priority job is always available and is executed at a given slot if no priority job is available.
Let $T$ be the time of the first idle slot. Then

$$T \sim \text{geometric}(1 - p)$$

Let $B$ be the length of the first busy period. Then

$$B \sim \text{geometric}(1 - p)$$

Let $I$ be the length of the first idle period. Then

$$I \sim \text{geometric}(p)$$

Let $Z$ be the number of slots after the first slot of the first busy period, including the first subsequent idle slot. Then

$$Z = B \implies Z \sim \text{geometric}(1 - p)$$
Example 6.3

Suppose $X \sim \text{BerPro}(p)$. Let $N$ be the first time that an arrival immediately follows an arrival. What is the probability

$$P(X_{N+1} = 0 \cap X_{N+2} = 0)$$

that there are no arrivals in the next two time slots?

$N$ depends on $X_{\leq N}$, so the restarted process

$$X' : X_{N+1}, X_{N+2}, \ldots$$

is a Bernoulli process $X' \sim \text{BerPro}(p)$, and

$$P(X_{N+1} = 0 \cap X_{N+2} = 0) = P(X'_1 = 0 \cap X'_2 = 0) = (1 - p)^2$$
The time of an arrival is called an **arrival time**. The time between two arrivals is called an **inter-arrival time**.

**Relation.** Let $Y_k$ be the **arrival time** of the $k$th arrival of an arrival process, and $T_k$ be the **inter-arrival time** between arrival $(k - 1)$ and arrival $k$. Then

$$Y_k = T_1 + \cdots + T_k \quad \text{and} \quad T_k = Y_k - Y_{k-1}$$
The inter-arrival times of a Bernoulli process are **i.i.d.** geometric random variables.

**Explanation.** Let \( X \sim \text{BerPro}(p) \). The first arrival time \( T_1 \) is geometric, with

\[
T_1 \sim \text{geometric}(p)
\]

Let \( X' \) be \( X \) re-started at \( T_1 \). Then \( X' \sim \text{BerPro}(p) \) by the fresh-start property. The inter-arrival time \( T_2 \) of \( X \) is the first arrival time \( Y'_1 \) of \( X' \), so

\[
T_2 = Y'_1 \sim \text{geometric}(p)
\]
Example 6.4

It has been observed that after a rainy day, the number of days until it rains again is geometrically distributed with parameter $p$, independent of the past. Find the probability that it rains on both the 5th and the 8th day of the month.

$$P(X_5 = 1 \cap X_8 = 1) = p^2$$
An arrival time of a Bernoulli process has a Pascal PMF.

**Derivation.** Let $Y_k$ be the $k$th arrival time of $X \sim \text{BerPro}(p)$. Event $Y_k = n$ means that there are $k-1$ arrivals from time 1 to time $n-1$ and an arrival at time $n$. It also requires $n \geq k$. Thus

$$p_{Y_k}(n) = \binom{n-1}{k-1} p^{k-1} (1-p)^{n-1-(k-1)} \times \frac{p}{n}\binom{n}{k-1} p^k (1-p)^{n-k}, \quad n \geq k$$

This is called the Pascal PMF.
Example 6.5

In each playing minute, Lin commits a foul with probability $p$ and no foul with probability $1 - p$. Committed fouls are independent. Lin plays 30 minutes if he does not foul out. What is the PMF of Lin’s playing time $Z$?

Lin’s fouls can be seen as arrivals of $X \sim \text{BerPro}(p)$. Let $Y_k$ be the $k$th arrival time of $X$. Then

$$Z = \min(Y_6, 30)$$

We have equivalent events

$$(Z = 30) = (Y_6 \geq 30), \quad (Z = n) = (Y_6 = n), \quad n = 6, \ldots, 29$$

so

$$p_Z(n) = \begin{cases} 
    p_{Y_6}(n), & 6 \leq n \leq 29 \\
    1 - \sum_{n' = 6}^{29} p_{Y_6}(n'), & n = 30 
\end{cases}$$
Splitting a Bernoulli process produces Bernoulli processes.

Let $X \sim \text{BerPro}(p)$ be split stochastically: an arrival is **accepted** with probability $q$, and **not accepted** with probability $1 - q$. Then the accepted arrivals of $X$ is an arrival process

$$Y \sim \text{BerPro}(pq)$$

The unaccepted arrivals of $X$ is another arrival process

$$Z \sim \text{BerPro}(p(1 - q))$$
Merging Bernoulli processes produces a Bernoulli process.

Let $W$ be the merged process of $X \sim \text{BerPro}(p)$ and $Y \sim \text{BerPro}(q)$: an arrival of $X$ or $Y$ is an arrival of $W$. Then

$$W \sim \text{BerPro}(p + q - pq)$$
Poisson Processes
A Poisson process is a continuous-time arrival process with uniform and independent arrivals.

Let $N(\delta)$ be the number of arrivals of a Poisson process in a very small period of length $\delta$. By assumption

$$P(N(\delta) = k) = \begin{cases} 
1 - \lambda \delta + o(\delta), & k = 0 \\
\lambda \delta + o(\delta), & k = 1 \\
o(\delta), & k > 1
\end{cases}$$

where $\lambda > 0$ is a parameter.
Arrival Rate

The parameter $\lambda$ of a Poisson process is the expected number of arrivals per unit time, i.e. arrival rate.

**Proof.** The expected number of arrivals in a very small period is

$$E[N(\delta)] = \lambda \delta + o(\delta)$$

Thus

$$\lim_{\delta \to 0^+} \frac{E[N(\delta)]}{\delta} = \lambda$$

A Poisson process with arrival rate $\lambda$ is denoted by

$$X \sim \text{PoiPro}(\lambda)$$
Arrivals in a Finite Period

The number of arrivals of in a Poisson process in a finite period is a Poisson random variable.

**Explanation.** Consider a Poisson process with arrival rate $\lambda$. For a very small period, the number of arrivals $N(\delta)$ is approximately

$$N(\delta) \sim \text{Bernoulli}(\lambda \delta)$$

A finite period of length $\tau$ can be partitioned into $n \gg 1$ disjoint periods of small length $\left(\frac{\tau}{n}\right)$. Then $N(\tau)$ is the sum of $n$ Bernoulli random variables with parameter $\lambda \left(\frac{\tau}{n}\right)$

$$N(\tau) \sim \text{binomial} \left(n, \lambda \left(\frac{\tau}{n}\right)\right) \approx \text{Poisson} \left(n \lambda \left(\frac{\tau}{n}\right)\right) = \text{Poisson}(\lambda \tau)$$
<table>
<thead>
<tr>
<th>Number of periods:</th>
<th>Probability of success per period:</th>
<th>Expected number of arrivals:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = \tau / \delta$</td>
<td>$p = \lambda \delta$</td>
<td>$np = \lambda \tau$</td>
</tr>
</tbody>
</table>

![Arrival Process Diagram](image-url)
Example 6.8

Bill gets e-mails according to a Poisson process at an arrival rate of $\lambda = 0.2$ messages per hour. He checks email every hour. What is the probability of no new message? 1 new message?

\[ N(1) \sim \text{Poisson}(0.2 \times 1) \]
Arrivals of customers at a local supermarket are modeled by a Poisson process with an arrival rate of $\lambda = 10$ customers per minute. $M$ is the number of customers arriving between 9:00 and 9:10, and $N$ is the number of customers arriving between 9:30 and 9:35. What is the PMF of $M + N$?

$$(M + N) \sim (\text{Poisson}(10 \times 10) + \text{Poisson}(10 \times 5)) = \text{Poisson}(150)$$
The first arrival time of a Poisson process is an exponential random variable.

**Explanation.** Consider a Poisson process with arrival rate $\lambda$. The key is to note the following equivalent events

$$(Y_1 > t) = \{\text{no arrivals in } (0, t)\} = (N(t) = 0)$$

They have the same probability, so

$$F_{Y_1}(t) = P(Y_1 \leq t) = 1 - P(Y_1 > t) = 1 - P(N(t) = 0)$$

$$= 1 - e^{-\lambda t}$$

That is

$$Y_1 \sim \text{exponential}(\lambda)$$
A re-started Poisson process is a Poisson process.

Consider a Poisson process with arrival rate $\lambda$.

- Suppose the process is re-started at $u \geq 0$. The restarted process $X' : X'_t = X_{u+t}$ is $\text{PoiPro}(\lambda)$.

- Suppose the process is re-started at a random time $U \geq 0$. If $U$ depends only on $\{X_t, t \leq U\}$, then the re-started process $X' : X'_t = X_{U+t}$ is $\text{PoiPro}(\lambda)$.
Example 6.10

You and your partner go to a badminton court, and have to wait until the players in the court finish playing. Assume that their playing time has an exponential PDF. Then the PDF of your waiting time has the same exponential PDF, regardless of when they started playing.
Example 6.11

When you enter the bank, you find all 3 tellers are busy serving customers, and there are no other customers in queue. After you, no more customers are allowed to enter. Assume that the service times for you and for each of the customers being served are i.i.d. exponential random variables. What is the probability that you will be the last to leave?

\[
\frac{1}{4} \text{? } \frac{1}{3}?
\]
An arrival time of a Poisson process has an **Erlang PDF**.

**Derivation.** Let $Y_k$ be the $k$th arrival time of a Poisson process with arrival rate $\lambda$. Then

\[
P(t < Y_k < t + \delta) = P(N(t) = k - 1) \times P(N(\delta) = 1)
\]

\[
= \frac{(\lambda t)^{k-1}}{(k - 1)!} e^{-\lambda t} \times (\lambda \delta + o(\delta))
\]

\[
= F_{Y_k}(t + \delta) - F_{Y_k}(t)
\]

Thus

\[
f_{Y_k}(t) = \lim_{\delta \to 0^+} \frac{F_{Y_k}(t + \delta) - F_{Y_k}(t)}{\delta} = \frac{\lambda^k t^{k-1}}{(k - 1)!} e^{-\lambda t}
\]
Example 6.12

You call the IRS hotline. You are the 56th person waiting to be served. Callers depart according to a Poisson process with a rate of \( \lambda = 2 \) per minute. How long will you have to wait on average until your service starts, and what is the probability that you will have to wait for more than 30 minutes?

Your wait until everyone in front of you have departed, so your waiting time is

\[
W = Y_{56}
\]

\[
E[W] = E[Y_{56}] = \frac{56}{2} = 28
\]

\[
P(W > 30) = P(Y_{56} > 30) = \int_{30}^{\infty} \frac{2^{56}t^{55}}{(55)!} e^{-2t} dt
\]

Exercise: Estimate the probability by CLT.
PDFs of Inter-arrival Times

The inter-arrival times of a Poisson process are i.i.d. exponential variables.

**Application.** A sample path of a Poisson process with rate $\lambda$ can be generated as follows.

- Sample inter-arrival times

  $$T_i \sim \text{exponential}(\lambda)$$

- Set $X_t$ to 1 at

  $$t = T_1, T_1 + T_2 \ldots$$

  Set $X_t$ to 0 otherwise.
Splitting a Poisson process produces Poisson processes.

Let $X \sim \text{PoiPro}(\lambda)$ be split stochastically: an arrival is accepted with probability $q$, and not accepted with probability $1 - q$. The accepted arrivals define an arrival process $Y \sim \text{PoiPro}(\lambda q)$ with

\[
P(N(\delta) = 1) = [\lambda \delta + o(\delta)]q = (\lambda q)\delta + o(\delta)
\]
\[
P(N(\delta) = 0) = [1 - \lambda \delta + o(\delta)] \cdot 1 + [\lambda \delta + o(\delta)] \cdot (1 - q) = 1 - (\lambda q)\delta + o(\delta)
\]
\[
P(N(\delta) = n) = o(\delta), \quad n \geq 2
\]

The unaccepted arrivals define another arrival process

$Z \sim \text{PoiPro}(\lambda(1 - q))$
Merging Poisson processes produces a Poisson process.

Let $W$ be the merged process of $X \sim \text{PoiPro}(\lambda_1)$ and $Y \sim \text{PoiPro}(\lambda_2)$: an arrival of $X$ or $Y$ is an arrival of $W$. Then

$$W \sim \text{PoiPro}(\lambda_1 + \lambda_2)$$

**Explanation.** Consider the arrivals of $W$ in a very small period.

$$P(N(\delta) = 0) = (1 - \lambda_1 \delta + o(\delta))(1 - \lambda_2 \delta + o(\delta))$$

$$= 1 - (\lambda_1 + \lambda_2)\delta + o(\delta)$$

$$P(N(\delta) = 1) = \lambda_1 \delta(1 - \lambda_2 \delta) + (1 - \lambda_1 \delta)(\lambda_2 \delta) + o(\delta)$$

$$= (\lambda_1 + \lambda_2)\delta + o(\delta)$$

So

$$W \sim \text{PoiPro}(\lambda_1 + \lambda_2)$$
Example 6.13

The arrivals of packets at a node of a data network define a Poisson process with rate $\lambda$. Each packet is either a local packet (with probability $p$) or a transit packet (with probability $1 - p$), independent of other arrivals and independent of the arrival times.

The arrival of local packets is a Poisson process with rate $\lambda p$, i.e.

$$\text{PoiPro}(\lambda p)$$
Example 6.14

At a post office, the arrivals of people with letters to mail define a Poisson process with rate $\lambda_1$, and the arrivals of people with packages to mail define a Poisson process with rate $\lambda_2$. The arrivals are independent.

The arrivals of people with letters or packages to mail define a Poisson process

$$W \sim \text{PoiPro}(\lambda_1 + \lambda_2)$$

A person arrives at the post office. He wants to mail a letter with probability

$$\frac{\lambda_1}{\lambda_1 + \lambda_2}$$
Example 6.15

Two light bulbs have lifetimes $T_a$ and $T_b$ which are independent exponential random variables with parameters $\lambda_a$ and $\lambda_b$. What is the distribution for the first time a bulb burns out?

$$Y_1 = \min(T_a, T_b) \quad \Rightarrow \quad (Y_1 > t) = (T_a > t \cap T_b > t)$$

$$\Rightarrow \quad P(Y_1 > t) = P(T_a > t) \cdot P(T_b > t)$$

$$\Rightarrow \quad 1 - P(Y_1 < t) = e^{-\lambda_a t} e^{-\lambda_b t}$$

$$\Rightarrow \quad P(Y_1 < t) = 1 - e^{-(\lambda_a + \lambda_b) t} = F_{Y_1}(t)$$

So

$$f_{Y_1}(t) = (\lambda_a + \lambda_b) e^{-(\lambda_a + \lambda_b) t}$$

That is

$$Y_1 \sim \text{exponential}(\lambda_a + \lambda_b)$$
Example 6.16

Three light bulbs have lifetimes which are independent exponential random variables with a common parameter \( \lambda \). What is the expected value of the time until the last bulb burns out?

\[
E[Y_1^{(3)} + Y_1^{(2)} + Y_1^{(1)}] = E[Y_1^{(3)}] + E[Y_1^{(2)}] + E[Y_1^{(1)}]
\]

\[
= \frac{1}{3\lambda} + \frac{1}{2\lambda} + \frac{1}{\lambda}
\]
Random Sum
Certain random sums are related to Bernoulli process and Poisson process.

**Definition.** Recall that a random sum is

\[ X_1 + \cdots + X_N \]

where \( X_1, X_2, \ldots \) are i.i.d. random variables and the number of terms \( N \) is a random variable independent of every \( X_i \).
A geometric random sum of i.i.d. geometric random variables is a geometric random variable.

If

\[ X_i \sim \text{geometric}(p), \ N \sim \text{geometric}(q) \]

then

\[ (X_1 + \cdots + X_N) \sim \text{geometric}(pq) \]
The above random sum is the first arrival time of a split Bernoulli process.

Split $Z \sim \text{BerPro}(p)$ with probability $q$ to get $Z' \sim \text{BerPro}(pq)$. Let $X_i$ be the $i$th inter-arrival time of $Z$, $N$ be the number of arrivals in $Z$ until an arrival is accepted by $Z'$, and $Y'_1$ be the first arrival time of $Z'$. Then $Y'_1 = X_1 + \cdots + X_N$. From

$$X_i \sim \text{geometric}(p), \quad N \sim \text{geometric}(q), \quad Y'_1 \sim \text{geometric}(pq)$$

we get

$$(X_1 + \cdots + X_N) \sim \text{geometric}(pq)$$
A geometric random sum of i.i.d. exponential random variables is an exponential random variable.

If

\[ X_i \sim \text{exponential}(\lambda), \; N \sim \text{geometric}(q) \]

then

\[ (X_1 + \cdots + X_N) \sim \text{exponential}(\lambda q) \]
The above random sum is the first arrival time of a split Poisson process.

Split $W \sim \text{PoiPro}(\lambda)$ with probability $q$ to get $W' \sim \text{PoiPro}(\lambda q)$. Let $X_i$ be the $i$th inter-arrival time of $W$, $N$ be the number of arrivals in $W$ until an arrival is accepted by $W'$, and $Y_1'$ be the first arrival time of $W'$. Then $Y_1' = X_1 + \cdots + X_N$. From

$$X_i \sim \text{exponential}(\lambda), \ N \sim \text{geometric}(q), \ Y_1' \sim \text{exponential}(\lambda q)$$

we get

$$(X_1 + \cdots + X_N) \sim \text{exponential}(\lambda q)$$
A **binomial** random sum of i.i.d. **Bernoulli** random variables is a **binomial** random variable.

If

\[ X_i \sim \text{Bernoulli}(q), \quad N \sim \text{binomial}(m, p) \]

then

\[ (X_1 + \cdots + X_N) \sim \text{binomial}(m, pq) \]
The above random sum is the number of arrivals of a split Bernoulli process.

Split $Y \sim \text{BerPro}(p)$ with probability $q$ to get $Y' \sim \text{BerPro}(pq)$. Let $X_i$ be a Bernoulli random variable such that $X_i = 1$ if the $i$th arrival of $Y$ is accepted by $Y'$, and $X_i = 0$ otherwise. Let $N$ be the number of arrivals in $Y$ during the first $m$ slots. Let $N_a$ be the number of arrivals in $Y'$ during the first $m$ slots. Then

$$N_a = X_1 + \cdots + X_N.$$ 

From

$$X_i \sim \text{Bernoulli}(q), \ N \sim \text{binomial}(m, p), \ N_a \sim \text{binomial}(m, pq)$$

we get

$$(X_1 + \cdots + X_N) \sim \text{binomial}(m, pq)$$
A Poisson random sum of i.i.d. Bernoulli random variables is a Poisson random variable.

If

\[ X_i \sim \text{Bernoulli}(q), \quad N \sim \text{Poisson}(\lambda) \]

then

\[ (X_1 + \cdots + X_N) \sim \text{Poisson}(\lambda q) \]
The above random sum is the number of arrivals of a split Poisson process.

Split $W \sim \text{PoiPro}(\lambda)$ with probability $q$ to get $W' \sim \text{PoiPro}(\lambda q)$. Let $X_i$ be a Bernoulli random variable that indicates whether the $i$th arrival of $W$ is accepted by $W'$. Let $N$ be the number of arrivals in $W$ during $[0, 1]$. Let $N_a$ be the number of arrivals in $W'$ during $[0, 1]$. Then $N_a = X_1 + \cdots + X_N$. From

$$X_i \sim \text{Bernoulli}(q), \ N \sim \text{Poisson}(\lambda), \ N_a \sim \text{Poisson}(\lambda q)$$

we get

$$(X_1 + \cdots + X_N) \sim \text{Poisson}(\lambda q)$$
Bernoulli process with arrival probability $p$

$$X \sim \text{BerPro}(p), \quad X_n \sim \text{Bernoulli}(p)$$

Number of arrivals

$$S_n \sim \text{binomial}(n, p)$$

Interarrival time

$$T_n \sim \text{geometric}(p)$$

Splitting and merging

$$Y \sim \text{BerPro}(pq), \quad W \sim \text{BerPro}(p + q - pq)$$
**Summary: Poisson Process**

- Poisson process with arrival rate $\lambda$
  
  $$X \sim \text{PoiPro}(\lambda), \ N(\delta) \approx \text{Bernoulli}(\lambda\delta)$$

- # of arrivals
  
  $$N(\tau) \sim \text{Poisson}(\lambda\tau)$$

- Interarrival time
  
  $$T_n \sim \text{exponential}(\lambda)$$

- Splitting and merging
  
  $$Y \sim \text{PoiPro}(\lambda p), \ W \sim \text{PoiPro}(\lambda_1 + \lambda_2)$$