Basic Arrival Processes

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Probability

Outline

- Random process
- Arrival process
- Bernoulli process
- Poisson process
- Random sum and arrival process

Bernoulli Processes

Random process

Let $(\Omega, \mathcal{F}, P)$ be a probability model. A random process defined on Ω, say $\boldsymbol{X}$, has the following property.

- $\boldsymbol{X}$ contains random variables, each defined on Ω
- A random variable of $\boldsymbol{X}$ has an index
- X_t is the random variable with index t
- $\boldsymbol{X}$ is discrete-time if the set of index is discrete
- $\boldsymbol{X}$ is continuous-time if the set of index is continuous
- An ω in Ω is mapped to an instance of $\boldsymbol{X}$ called sample sequence or sample function

Arrival process

Let $(\Omega, \mathcal{F}, P)$ be a probability model. An arrival process defined on Ω, say $\boldsymbol{X}$, has the following property.

- Every X_t is a Bernoulli random variable
- $X_t = 1$ for an arrival at t and $X_t = 0$ for no arrival at t

We can use arrival processes for 2-state phenomenon.

- speech activity
- anomalous sound detection
- virus screening

Definition (Bernoulli process)

Let $(\Omega, \mathcal{F}, P)$ be a probability model. A Bernoulli process, say $\boldsymbol{X}$, has the following property.

- $\boldsymbol{X}$ is a discrete-time arrival process
- X_t indicates whether there is an arrival at epoch t
- X_t's are **iid** Bernoulli random variables
- Since every X_t is **Ber**(p) for some p, we can denote $\boldsymbol{X}$ by

  Bernoulli(p)

Example (6.1 Bernoulli process)

Let $\boldsymbol{X}$ be a Bernoulli process.

- Let U be the arrival count of $\boldsymbol{X}$ from time 1 to 5 and V be the arrival count of $\boldsymbol{X}$ from time 6 to 10. We have $U \perp\!\!\!\perp V$.
- Let W be the first odd time index with an arrival of $\boldsymbol{X}$ and Y be the first even time index with an arrival of $\boldsymbol{X}$. We have $W \perp\!\!\!\perp Y$.

Binomial arrivals and geometric arrival time

Let $\boldsymbol{X}$ be **Bernoulli**(p).

- Let S_n be the arrival count of $\boldsymbol{X}$ from time 1 to time n.

$$S_n \sim \textbf{Bin}(n, p)$$

- Let T be the first arrival time of $\boldsymbol{X}$.

$$T \sim \textbf{Geo}(p)$$

- We have $S_n = X_1 + \cdots + X_n$ where X_i's are **iid Ber**(p). So

$$S_n \sim \textbf{Bin}(n, p)$$

- The PDF of T is

$$\begin{aligned} P(T = n) &= P((X_1 = 0) \cap \cdots \cap (X_{n=1} = 0))P(X_n = 1) \\ &= (1-p)^{n-1} p \end{aligned}$$

Fresh-start and memoryless property

Let $\boldsymbol{X}$ be **Bernoulli**(p).

- Let n be a non-negative integer. Let $\boldsymbol{X}'$ be the part of $\boldsymbol{X}$ discarding $X_1, \ldots, X_n$, i.e.

  $$X'_t = X_{n+t}, \; t = 1, 2, \cdots$$

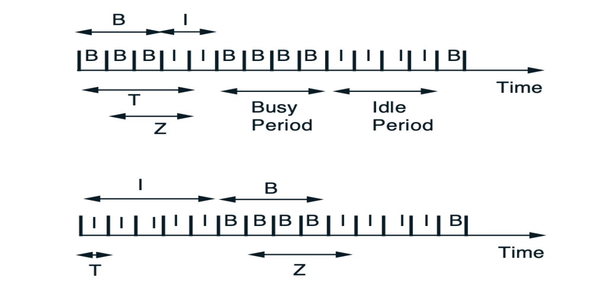
  Then $\boldsymbol{X}'$ is **Bernoulli**(p).
- Let N be a non-negative integer random variable that is independent of $X_{N+1}, X_{N+2}, \ldots$. Let $\boldsymbol{X}''$ be part of $\boldsymbol{X}$ discarding $X_1, \ldots, X_N$, i.e.

  $$X''_t = X_{N+t}, \; t = 1, 2, \cdots$$

  Then $\boldsymbol{X}''$ is **Bernoulli**(p).

Example (6.2)

A time-slotted computer executes a **priority** job with probability p in each slot, independent of other slots. A **slot** is **busy** if the computer executes a priority job, **idle** otherwise. A string of idle slots, flanked by busy slots, is an **idle period**. A string of busy slots, flanked by idle slots, is a **busy period**. Let's look at the random variables T, B, I, Z as shown in the following figure.



Random variables

- T is the time of the first idle slot. Since a slot is idle with probability $(1-p)$, T is **Geo**$(1-p)$.
- B is the length of the first busy period. Let $\boldsymbol{X}$ be **Bernoulli**(p), treating busy slots as arrivals. Let N be the time of the first busy slot. The first busy period ends as soon as an idle slot arrives after N. Let $\boldsymbol{X}'$ be the part of $\boldsymbol{X}$ discarding $X_1, \ldots, X_N$. Then $\boldsymbol{X}'$ is **Bernoulli**(p). Since B equals the time of the first idle slot of $\boldsymbol{X}'$, it is **Geo**$(1-p)$.
- I is the length of the first idle period. This period ends when a busy slot arrives after the first idle slot. So I is **Geo**(p).
- Z is the time of the first idle slot after the first busy slot. Since $Z = B - 1 + 1 = B$, we have $Z \sim$ **Geo**$(1-p)$.

Example (6.3)

Let $\boldsymbol{X}$ be **Bernoulli**(p). Let N be the first time that an arrival of $\boldsymbol{X}$ immediately follows the previous arrival of $\boldsymbol{X}$. What is the probability of no arrivals in the next two time slots, i.e.

$$P(X_{N+1} = 0 \cap X_{N+2} = 0)$$

N is independent of $X_{N+1}, X_{N+2}, \ldots$. Let $\boldsymbol{X}'$ be the part of $\boldsymbol{X}$ discarding $X_1, \ldots, X_N$, i.e.

$$X'_t = X_{N+t},\ t = 1, 2, \cdots$$

Then $\boldsymbol{X}'$ is **Bernoulli**(p). Thus

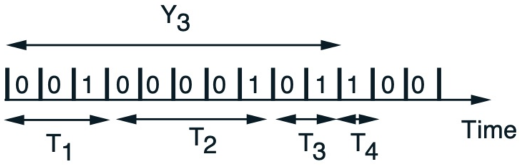
$$\begin{aligned} P(X_{N+1} = 0 \cap X_{N+2} = 0) &= P(X'_1 = 0 \cap X'_2 = 0) \\ &= P(X'_1 = 0)\, P(X'_2 = 0) \\ &= (1-p)^2 \end{aligned}$$

Definition (Arrival time, inter-arrival time)

Let $\boldsymbol{X}$ be an arrival process.

- The time of an arrival of $\boldsymbol{X}$ is an arrival time
- The time between an arrival and the previous arrival of $\boldsymbol{X}$ is an interarrival time
- Let Y_k be the time of the kth arrival of $\boldsymbol{X}$, and T_k be the time between the $(k-1)$th arrival and the kth arrival of $\boldsymbol{X}$. Then

$$Y_k = T_1 + \cdots + T_k$$



Inter-arrival times

Let $\boldsymbol{X}$ be **Bernoulli**(p) and T_k be the kth interarrival times of $\boldsymbol{X}$. Then $T_1, T_2, \cdots$ are **iid Geo**(p) random variables.

From $T_1 = Y_1$ and $Y_1 \sim$ **Geo**(p), we have

$$T_1 \sim \textbf{Geo}(p)$$

Let $\boldsymbol{X}'$ be $\boldsymbol{X}$ re-started at T_1 and Y_1' be the first arrival time of $\boldsymbol{X}'$. By the memoryless property, $\boldsymbol{X}'$ is **Bernoulli**(p) and Y_1' is **Geo**(p). From $T_2 = Y_1'$, we have

$$T_2 \sim \textbf{Geo}(p)$$

By same argument, T_{k+1} is the first arrival time of a restarted process of $\boldsymbol{X}$ restarted at $N = Y_k$, so T_{k+1} is **Geo**(p).

Example (6.4)

It has been observed that after a rainy day, the number of days until it rains again is a geometric random variable with parameter p, independent of the past. Find the probability that it rains on both the 5th and the 8th day of the month.

Let $\boldsymbol{X}$ be an arrival process, treating rainy days as arrivals. The **iid** inter-arrival times **Geo**(p) implies $\boldsymbol{X}$ is **Bernoulli**(p). Thus

$$\begin{aligned} P(X_5 = 1 \cap X_8 = 1) &= P(X_5 = 1)\, P(X_8 = 1) \\ &= p^2 \end{aligned}$$

Arrival times

Let $\boldsymbol{X}$ be **Bernoulli**(p) and Y_k be the kth arrival time of $\boldsymbol{X}$.

- The PMF of Y_k can be derived
- It is the Pascal PMF of order k with parameter p

Event $(Y_k = n)$ occurs if and only if that there are $(k-1)$ arrivals from time 1 to time $(n-1)$ and an arrival at time n. Thus

$$
\begin{aligned}
p_{Y_k}(n) &= \overbrace{\binom{n-1}{k-1} p^{k-1}(1-p)^{n-1-(k-1)}}^{(k-1) \text{ arrivals from time } 1 \text{ to time } (n-1)} \times \overbrace{p}^{\text{an arrival at time } n} \\
&= \begin{cases} \binom{n-1}{k-1} p^k (1-p)^{n-k}, & n = k, k+1, \ldots \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

Example (6.5)

In every minute Lin plays, he commits a foul with probability p. In a game, he plays until he's fouled out or he plays 30 minutes at most. What is the PMF of his playing time Z in a game?

Let $\boldsymbol{X}$ be **Bernoulli**(p), treating fouls as arrivals. Let Y_k be the kth arrival time of $\boldsymbol{X}$. Then we have $Z = \min(Y_6, 30)$ and

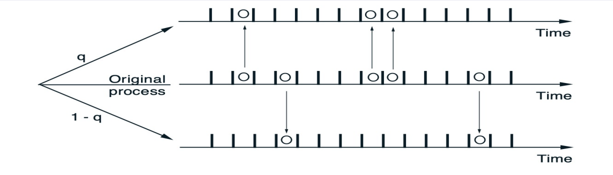
$$(Z = n) = \begin{cases} (Y_6 = n), & 6 \le n \le 29 \\ (Y_6 \ge 30), & n = 30 \end{cases}$$

Hence

$$p_Z(n) = \begin{cases} p_{Y_6}(n), & 6 \le n \le 29 \\ 1 - \sum\limits_{n'=6}^{29} p_{Y_6}(n'), & n = 30 \\ 0, & \text{otherwise} \end{cases}$$

Definition (Split process)

Let $\boldsymbol{X}$ be **Bernoulli**(p) and $\boldsymbol{W}, \boldsymbol{Z}$ be defined as follows: an arrival of $\boldsymbol{X}$ is an arrival of $\boldsymbol{W}$ with probability q, otherwise it is an arrival of $\boldsymbol{Z}$.



- We have $P(W_t = 1) = P(X_t = 1) \cdot q = pq$, so

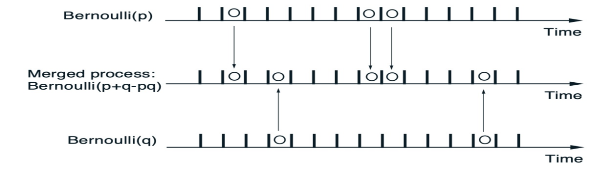
$$W_t \sim \textbf{Ber}(pq) \;\Rightarrow\; \boldsymbol{W} \sim \textbf{Bernoulli}(pq)$$

- Similarly

$$\boldsymbol{Z} \sim \textbf{Bernoulli}(p(1-q))$$

Definition (Merged process)

Let $\boldsymbol{X}$ be **Bernoulli**(p), $\boldsymbol{Z}$ be **Bernoulli**(q) and $\boldsymbol{X} \perp\!\!\!\perp \boldsymbol{Z}$. Let an arrival of $\boldsymbol{X}$ or an arrival of $\boldsymbol{Z}$ be an arrival of $\boldsymbol{W}$.



We have

$$
\begin{aligned}
P(W_t = 1) &= P((X_t = 1) \cup (Z_t = 1)) \\
&= P(X_t = 1) + P(Z_t = 1) - P((X_t = 1) \cap (Z_t = 1)) \\
&= p + q - pq
\end{aligned}
$$

So W_t's are **iid Ber**$(p+q-pq)$, and $\boldsymbol{W}$ is **Bernoulli**$(p+q-pq)$.

Poisson Processes

Definition (Poisson process)

Let $(\Omega, \mathcal{F}, P)$ be a probability model. A Poisson process $\boldsymbol{X}$ defined on Ω has the following property.

- $\boldsymbol{X}$ is a continuous-time arrival process
- In any period, the number of arrivals is Poisson RV
- Arrivals in non-overlapping periods are independent

Definition (Rate)

A Poisson process has an arrival rate. Specifically, a Poisson process $\boldsymbol{X}$ with rate λ has

$$N(t) \sim \textbf{Poi}(\lambda t)$$

where $N(t)$ is the number of arrivals in a period of length t.

This is denoted by

$$\boldsymbol{X} \sim \textbf{Poisson}(\lambda)$$

Small-interval arrivals

Let $\boldsymbol{X} \sim$ **Poisson**(λ) and $N(\delta)$ be the arrivals of $\boldsymbol{X}$ in a small period of length $0 < \delta \ll 1$. Since $N(\delta) \sim$ **Poi**$(\lambda\delta)$, we have

$$P(N(\delta) = k) = e^{-\lambda\delta} \frac{(\lambda\delta)^k}{k!}$$
$$= \begin{cases} 1 - \lambda\delta + o(\delta), & k = 0 \\ \lambda\delta + o(\delta), & k = 1 \\ o(\delta), & k > 1 \end{cases}$$

That is

$$N(\delta) \dot{\sim} \textbf{Ber}(\lambda\delta)$$

Arrival rate

Let $\boldsymbol{X} \sim$ **Poisson**(λ).

- The expected number of arrivals in a period of length τ is

$$\mathbf{E}[N(\tau)] = \lambda\tau$$

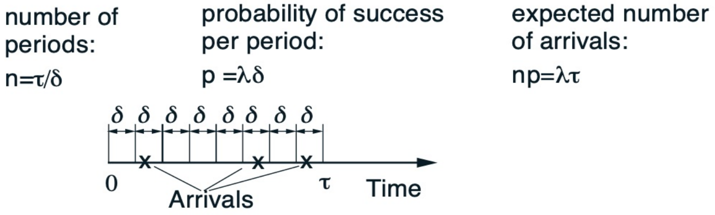
- The rate of arrival is

$$\frac{\mathbf{E}[N(\tau)]}{\tau} = \lambda$$

Small interval and finite interval

Let $\boldsymbol{X} \sim$ **Poisson**(λ). Let $N(\tau)$ be the number of arrivals in a period of length τ. Partition the period into small periods of length $\delta = \frac{\tau}{n}$ with $n \gg 1$. Let N_i be the arrival count in the ith small period. We have $N_i \dot\sim \mathbf{Ber}(\lambda\delta)$ and $N(\tau) = N_1 + \cdots + N_n$. Hence

$$N(\tau) \dot\sim \mathbf{Bin}(n, \lambda\delta) \xrightarrow{n\to\infty} \mathbf{Poi}\,(n\lambda\delta) = \mathbf{Poi}\,(\lambda\tau)$$



Example (6.8)

Bill gets e-mails according to **Poisson**(λ) with $\lambda = 0.2$ messages per hour. He checks email every hour. What is the probability of no new message? 1 new message?

Since $N(\tau) \sim \textbf{Poi}\,(\lambda\tau)$, we have

$$N(1) \sim \textbf{Poi}(0.2 \times 1)$$

Example (6.9)

Arrivals of customers at a supermarket are modeled by **Poisson**(λ) with $\lambda = 10$ customers per minute. Let M (resp. N) be the number of the customers arriving between 9:00 and 9:10 (resp. between 9:30 and 9:35). What is the PMF of $M + N$?

We have $M \sim$ **Poi**$(\lambda\tau_1) =$ **Poi**$(10 \cdot 10)$, $N \sim$ **Poi**$(\lambda\tau_2) =$ **Poi**$(10 \cdot 5)$, and $M \perp\!\!\!\perp N$. The sum of 2 independent Poisson random variables is a Poisson random variable. Thus

$$(M + N) \sim \textbf{Poi}(100 + 50)$$

First arrival time

Let $\boldsymbol{X} \sim$ **Poisson**(λ). The time of the first arrival of $\boldsymbol{X}$ is **Exp**(λ).

Let Y_1 be the first arrival time of $\boldsymbol{X}$ and $N(t)$ be the number of the arrivals in $(0, t]$. Note the time-count duality $(Y_1 > t)$ if and only if $(N(t) = 0)$. So $P(Y_1 > t) = P(N(t) = 0)$ and

$$P(Y_1 \leq t) = 1 - P(Y_1 > t) = 1 - P(N(t) = 0)$$

Since $N(t) \sim$ **Poi**(λt), we have

$$P(N(t) = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t},\ k = 0, 1, 2, \ldots$$

So the CDF of Y_1 is

$$F_{Y_1}(t) = 1 - P(N(t) = 0) = 1 - e^{-\lambda t}$$

which is the CDF of **Exp**(λ). Therefore, $Y_1 \sim$ **Exp**(λ).

Fresh-start/memoryless property

Consider $\boldsymbol{X} \sim$ **Poisson**(λ).

- Let $u > 0$ and $\boldsymbol{X}'$ be the part of $\boldsymbol{X}$ discarding $\boldsymbol{X}_{\le u}$, i.e.

$$X'_t = X_{u+t},\ t > 0$$

  Thus $\boldsymbol{X}' \sim$ **Poisson**(λ).

- Let U be a non-negative random variable and be independent of $\boldsymbol{X}_{>U}$. Let $\boldsymbol{X}''$ be the part of $\boldsymbol{X}$ discarding $\boldsymbol{X}_{\le U}$, i.e.

$$X''_t = X_{U+t},\ t > 0$$

  Thus $\boldsymbol{X}'' \sim$ **Poisson**(λ).

Example (6.10)

You and a partner go to the gym to play badminton. You wait until the on-court players to finish. Assume that their playing time is **Exp**(λ). Then your waiting time is **Exp**(λ), regardless of how long they have been playing.

Imagine $\boldsymbol{X} \sim$ **Poisson**(λ) that starts at the same time as the players on court. Their playing time is **Exp**(λ), so it is the first arrival time of $\boldsymbol{X}$. Let $\boldsymbol{X}'$ be the process obtained from restarting $\boldsymbol{X}$ at the same time as you begin to wait. Then your waiting time is the first arrival time of $\boldsymbol{X}'$. Since $\boldsymbol{X}' \sim$ **Poisson**(λ), your waiting time is **Exp**(λ).

Example (6.11)

When you enter a bank, all 3 tellers are busy serving customers, and there are no other customers in queue. No more customers enter the bank after you. Assume that the service times for the bank customers are **iid** exponential random variables. What is the probability that you will be the last customer to leave the bank?

$$\frac{1}{4}?\ \frac{1}{3}?$$

Inter-arrival time

Let $\boldsymbol{X} \sim$ **Poisson**(λ) and T_k be the kth **interarrival time** of $\boldsymbol{X}$. Then $T_1, T_2, \cdots$ are **iid Exp**(λ) random variables.

- This follows from the memoryless property.
- That is, by restarting $\boldsymbol{X}$ at an arrival time, the next arrival time is **Exp**(λ) and is an interarrival time of $\boldsymbol{X}$.
- An instance of $\boldsymbol{X}$ can be generated as follows: for $k = 1, 2, \ldots$ sample interarrival time

$$T_k \sim \textbf{Exp}(\lambda)$$

and set X_t to 1 at **arrival time**

$$Y_k = T_1 + \cdots + T_k$$

Arrival times

Let $\boldsymbol{X} \sim$ **Poisson**(λ) and Y_k be the kth arrival time of $\boldsymbol{X}$.

- The PDF Y_k can be derived
- It is the Erlang PDF of order k with parameter λ

We have

$$\begin{aligned} P(Y_k \in (t, t+\delta)) &= \overbrace{P(N(t) = k-1)}^{(k-1) \text{ arrivals in } (0,t)} \times \overbrace{P(N(\delta) = 1)}^{1 \text{ arrival in } (t, t+\delta)} \\ &= \frac{(\lambda t)^{k-1}}{(k-1)!} e^{-\lambda t} \times (\lambda\delta + o(\delta)) \\ &= F_{Y_k}(t+\delta) - F_{Y_k}(t) \end{aligned}$$

So the PDF of Y_k is

$$f_{Y_k}(t) = \lim_{\delta \to 0^+} \frac{F_{Y_k}(t+\delta) - F_{Y_k}(t)}{\delta} = \frac{\lambda^k t^{k-1}}{(k-1)!} e^{-\lambda t}$$

Example (6.12)

You call IRS hotline and are the 56th person waiting to be served. Suppose the callers' departure is **Poisson**(λ) with $\lambda = 2$ per minute. How long do you expect to wait until your service starts, and what is the probability that the waiting time is more than 30 minutes?

Let W be the waiting time, $\boldsymbol{X} \sim$ **Poisson**(λ) be callers' departures, and $T_k \sim$ **Exp**(λ) be the kth interarrival time of $\boldsymbol{X}$. Then

$$W = T_1 + \cdots + T_{56} = Y_{56}$$

We have

$$\mathbf{E}[W] = \mathbf{E}[T_1 + \cdots + T_{56}] = \mathbf{E}[T_i] \cdot 56 = \frac{1}{\lambda} \cdot 56 = 28$$

and

$$P(W > 30) = P(Y_{56} > 30) = \int_{30}^{\infty} \frac{2^{56} t^{55}}{(55)!} e^{-2t} dt$$

Definition (Split process)

Consider $\boldsymbol{X} \sim$ **Poisson**(λ). Let the arrivals of $\boldsymbol{X}$ be split as follows. An arrival of $\boldsymbol{X}$ is an arrival of $\boldsymbol{W}$ with probability q, otherwise it is an arrival of $\boldsymbol{Z}$.

Let $N(\delta)$ be the number of arrivals of $\boldsymbol{W}$ in a period of length δ.

$$\begin{cases} P(N(\delta)=1) = [\lambda\delta + o(\delta)]q = (\lambda q)\delta + o(\delta) \\ P(N(\delta)=0) = [1-\lambda\delta+o(\delta)]\cdot 1 + [\lambda\delta + o(\delta)]\cdot(1-q) \\ \qquad\qquad\quad = 1-(\lambda q)\delta + o(\delta) \\ P(N(\delta)=k) = o(\delta), \quad k \geq 2 \end{cases}$$

So

$$\boldsymbol{W} \sim \textbf{Poisson}(\lambda q)$$

Similarly

$$\boldsymbol{Z} \sim \textbf{Poisson}(\lambda(1-q))$$

Definition (Merged process)

Let $\boldsymbol{X} \sim$ **Poisson**(λ_1) and $\boldsymbol{Z} \sim$ **Poisson**(λ_2). Let $\boldsymbol{W}$ be the process obtained by merging the arrivals of $\boldsymbol{X}$ and $\boldsymbol{Z}$.

Let $N(\delta)$ be the number of arrivals of $\boldsymbol{W}$ in a period of length δ.

$$
\begin{aligned}
P(N(\delta)=0) &= (1-\lambda_1\delta+o(\delta))(1-\lambda_2\delta+o(\delta)) \\
&= 1-(\lambda_1+\lambda_2)\delta+o(\delta) \\
P(N(\delta)=1) &= \lambda_1\delta(1-\lambda_2\delta)+(1-\lambda_1\delta)(\lambda_2\delta)+o(\delta) \\
&= (\lambda_1+\lambda_2)\delta+o(\delta) \\
P(N(\delta)=k) &= o(\delta), \quad k \geq 2
\end{aligned}
$$

So

$$\boldsymbol{W} \sim \textbf{Poisson}(\lambda_1+\lambda_2)$$

Example (6.13)

The arrivals of packets at a network node is modeled as **Poisson**(λ). An arrived packet is either a **local** packet with probability q or a **transit** packet with probability $1-q$, independent of other arrivals and independent of the arrival times. Then the arrivals of **local** packets is **Poisson**(λq). The arrivals of **transit** packets is **Poisson**$(\lambda(1-q))$.

Example (6.14)

Customers arrive at a post office according to **Poisson**(λ_1) to mail letters, or according to **Poisson**(λ_2) to mail packages.

- Regardless of letter or package, the arrive process of the customers is

$$\textbf{Poisson}(\lambda_1 + \lambda_2)$$

- A customer wants to mail a letter with probability

$$\frac{\lambda_1}{\lambda_1 + \lambda_2}$$

Example (6.15)

Let the lifetimes of two light bulbs $T_a \sim \textbf{Exp}(\lambda_a)$ and $T_b \sim \textbf{Exp}(\lambda_b)$ be independent. Let T be the first time that a bulb burns out. What is the PDF of T?

$$
\begin{aligned}
T = \min(T_a, T_b) \;&\Rightarrow\; (T > t) = (T_a > t \cap T_b > t) \\
&\Rightarrow\; P(T > t) = P(T_a > t \cap T_b > t) \\
&\Rightarrow\; 1 - P(T < t) = e^{-\lambda_a t}\, e^{-\lambda_b t} \\
&\Rightarrow\; P(T < t) = 1 - e^{-(\lambda_a + \lambda_b)t} \\
&\Rightarrow\; F_T(t) = 1 - e^{-(\lambda_a + \lambda_b)t}
\end{aligned}
$$

So

$$T \sim \textbf{Exp}(\lambda_a + \lambda_b)$$

Note T is the first arrival time of **Poisson**$(\lambda_a + \lambda_b)$ and

$$\mathbf{E}[T] = (\lambda_a + \lambda_b)^{-1}$$

Example (6.16)

Let the lifetimes of three light bulbs be **iid** **Exp**(λ). What is the expected value of the time until all bulbs burn out?

Let T_k be the time between the $(k-1)$th burn-out and the kth burn-out. There are $3-(k-1)=4-k$ light bulbs during that period. From Example 6.15, we are merging $(4-k)$ **Poisson**(λ)'s, and the first arrival time of the merged process is

$$T_k \sim \textbf{Exp}((4-k)\lambda)$$

The time until all bulbs burn out is $T_1+T_2+T_3$, with expectation

$$\begin{aligned}\mathbf{E}[T_1+T_2+T_3] &= \mathbf{E}[T_1]+\mathbf{E}[T_2]+\mathbf{E}[T_3] \\ &= (3\lambda)^{-1}+(2\lambda)^{-1}+(\lambda)^{-1}\end{aligned}$$

Random Sum and Arrival Processes

Random sum

Let $X_1, X_2, \cdots$ be random variables and N is a non-negative integer random variable. We consider the random sum defined by

$$S = X_1 + \cdots + X_N$$

Random sum and split Bernoulli process

Let $\boldsymbol{Z}$ be a Bernoulli process and $\boldsymbol{Z}'$ be the Bernoulli process obtained from splitting the arrivals of $\boldsymbol{Z}$.

- The first arrival time of $\boldsymbol{Z}'$ can be seen as a random sum
- The arrival count of $\boldsymbol{Z}'$ can be seen as a random sum

Random sum and split Poisson process

Let $\boldsymbol{W}$ be a Poisson process and $\boldsymbol{W}'$ be obtained from splitting the arrivals of $\boldsymbol{W}$.

- The first arrival time of $\boldsymbol{W}'$ can be seen as a random sum
- The arrival count of $\boldsymbol{W}'$ can be seen as a random sum

Geometric geometric sum

Let $T_1, T_2, \ldots$ be **iid** $\textbf{Geo}(p)$ and N be $\textbf{Geo}(q)$. We have

$$(T_1 + \cdots + T_N) \sim \textbf{Geo}(pq)$$

Let $\boldsymbol{Z} \sim \textbf{Bernoulli}(p)$ and $\boldsymbol{Z}'$ be obtained from splitting the arrivals of $\boldsymbol{Z}$ with probability q. Then $\boldsymbol{Z}' \sim \textbf{Bernoulli}(pq)$. Let Y_1' be the first arrival time of $\boldsymbol{Z}'$. From the perspective of $\boldsymbol{Z}'$, we have

$$Y_1' \sim \textbf{Geo}(pq)$$

From the perspective of $\boldsymbol{Z}$, we have

$$Y_1' = T_1 + \cdots + T_N$$

where $N \sim \textbf{Geo}(q)$ is the arrival count of $\boldsymbol{Z}$ until the first arrival of $\boldsymbol{Z}'$ occurs and T_i is the ith interarrival time of $\boldsymbol{Z}$. Hence

$$(T_1 + \cdots + T_N) \sim \textbf{Geo}(pq)$$

Binomial Bernoulli sum

Let $X_1, X_2, \ldots$ be **iid Ber**(q) and N be **Bin**(m, p). We have

$$(X_1 + \cdots + X_N) \sim \textbf{Bin}(m, pq)$$

Let $\boldsymbol{Z} \sim$ **Bernoulli**(p) and $\boldsymbol{Z}'$ be obtained from splitting the arrivals of $\boldsymbol{Z}$ with probability q. Then $\boldsymbol{Z}' \sim$ **Bernoulli**(pq). Let N' be the arrival count of $\boldsymbol{Z}'$ in $[1, m]$. From the perspective of $\boldsymbol{Z}'$, we have

$$N' \sim \textbf{Bin}(m, pq)$$

From the perspective of $\boldsymbol{Z}$, we have

$$N' = X_1 + \cdots + X_N$$

where $N \sim$ **Bin**(m, p) is the arrival count of $\boldsymbol{Z}$ in $[1, m]$ and X_i indicates whether the ith arrival of $\boldsymbol{Z}$ is an arrival of $\boldsymbol{Z}'$. Hence

$$(X_1 + \cdots + X_N) \sim \textbf{Bin}(m, pq)$$

Geometric exponential sum

Let $T_1, T_2, \ldots$ be **iid Exp**(λ) and N be **Geo**(q). We have

$$(T_1 + \cdots + T_N) \sim \textbf{Exp}(\lambda q)$$

Let $\boldsymbol{W} \sim$ **Poisson**(λ) and $\boldsymbol{W}'$ be obtained from splitting the arrivals of $\boldsymbol{W}$ with probability q. Then $\boldsymbol{W}' \sim$ **Poisson**(λq). Let Y_1' be the first arrival time of $\boldsymbol{W}'$. From the perspective of $\boldsymbol{W}'$, we have

$$Y_1' \sim \textbf{Exp}(\lambda q)$$

From the perspective of $\boldsymbol{W}$, we have

$$Y_1' = T_1 + \cdots + T_N$$

where $N \sim$ **Geo**(q) is the arrival count of $\boldsymbol{W}$ until the first arrival of $\boldsymbol{W}'$ occurs and T_i is the ith interarrival time of $\boldsymbol{W}$. Hence

$$(T_1 + \cdots + T_N) \sim \textbf{Exp}(\lambda q)$$

Poisson Bernoulli sum

Let $X_1, X_2, \ldots$ be **iid Ber**(q) and N be **Poi**(λ). We have

$$(X_1 + \cdots + X_N) \sim \textbf{Poi}(\lambda q)$$

Let N' be the number of the arrivals of $\boldsymbol{W}'$ in $(0, 1)$. From the perspective of $\boldsymbol{W}'$, we have

$$N' \sim \textbf{Poi}(\lambda q)$$

From the perspective of $\boldsymbol{W}$, we have

$$N' = X_1 + \cdots + X_N$$

where $N \sim \textbf{Poi}(\lambda)$ is the arrival count of $\boldsymbol{W}$ within $(0, 1)$, and X_i indicates whether the ith arrival of $\boldsymbol{W}$ is an arrival of $\boldsymbol{W}'$. Hence

$$(X_1 + \cdots + X_N) \sim \textbf{Poi}(\lambda q)$$

Summary 1

Bernoulli process with arrival probability p

$$\boldsymbol{X} \sim \textbf{Bernoulli}(p), \; X_n \sim \textbf{Ber}(p)$$

# of arrivals

$$S_n \sim \textbf{Bin}(n, p)$$

Interarrival time

$$T_n \sim \textbf{Geo}(p)$$

Splitting/merging arrivals

$$\boldsymbol{Y} \sim \textbf{Bernoulli}(pq), \; \boldsymbol{W} \sim \textbf{Bernoulli}(p + q - pq)$$

Summary 2

Poisson process with arrival rate λ

$$\boldsymbol{X} \sim \textbf{Poisson}(\lambda), \ N(\delta) \approx \textbf{Ber}(\lambda\delta)$$

# of arrivals

$$N(\tau) \sim \textbf{Poi}(\lambda\tau)$$

Interarrival time

$$T_n \sim \textbf{Exp}(\lambda)$$

Splitting/merging arrivals

$$\boldsymbol{Y} \sim \textbf{Poisson}(\lambda p), \ \boldsymbol{W} \sim \textbf{Poisson}(\lambda_1 + \lambda_2)$$