Final exam

- Released on 2025.05.28
- Do not use electronic devices in exam
- Answers without due explanation/reasoning will not be graded
- The problems are roughly arranged in the chronological order
- Each problem is 10 points unless stated otherwise
- Reference

$X \sim \textbf{Ber}(p) \;\Rightarrow\; p_X(x) = p^x(1-p)^{1-x}$ for $x = 0, 1$

$X \sim \textbf{Bin}(n,p) \;\Rightarrow\; p_X(x) = \binom{n}{x} p^x(1-p)^{n-x}$ for $x = 0, 1, \dots, n$

$X \sim \textbf{Geo}(p) \;\Rightarrow\; p_X(x) = p(1-p)^{x-1}$ for $x = 1, 2, \dots$

$X \sim \textbf{Uni}[a,b] \;\Rightarrow\; p_X(x) = \dfrac{1}{b-a+1}$ for $x = a, a+1, \dots, b$

$X \sim \textbf{Poi}(\lambda) \;\Rightarrow\; p_X(x) = e^{-\lambda}\dfrac{\lambda^x}{x!}$ for $x = 0, 1, \dots$

$X \sim \textbf{Uni}(a,b) \;\Rightarrow\; f_X(x) = \dfrac{1}{b-a}$ for $a < x < b$

$X \sim \textbf{Exp}(\lambda) \;\Rightarrow\; f_X(x) = \lambda e^{-\lambda x}$ for $x > 0, \lambda > 0$

$X \sim \mathcal{N}(\mu, \sigma^2) \;\Rightarrow\; f_X(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ for $x \in \mathbb{R}$

$X \sim \textbf{Beta}(a,b) \;\Rightarrow\; f_X(x) = \dfrac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1}$ for $x \in (0,1), a > 0, b > 0$

$X \sim \textbf{Gamma}(a,\lambda) \;\Rightarrow\; f_X(x) = \dfrac{1}{\Gamma(a)}(\lambda x)^{a-1}\lambda e^{-\lambda x}$ for $x > 0, a > 0, \lambda > 0$

moment generating function (MGF): $M_X(t) = \mathbf{E}\left(e^{tX}\right)$

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1. Consider 6-door version of Monty-Hall problem. Suppose Guest chooses Door 1 and Host opens Door 5. Decide the (conditional) probability that car is behind Door 3.

2. Team A and Team B are asked to design a new product within a month. Suppose Team A is successful with probability 1/3, Team B is successful with probability 1/2, and both teams are successful with probability 1/4. What is the probability that Team B fails given Team A succeeds?

3. Find the expected number of flips with a fair coin for 3 consecutive Heads to occur.

4. Random number of students $N \sim \text{Poi}(50)$ are present in a probability class. Among the students in the class, each falls asleep with probability 0.7, independent of other students. Compute the variance of $(X - Y)$ where X is be the number of students that fall asleep and Y is the number of students that stay awake.

5. Let S be a continuous random variable with CDF

$$F_S(x) = \frac{1}{1 + e^{-2x}}, \quad x \in \mathbb{R}$$

Describe how one can obtain samples of S based on samples of $U \sim \textbf{Uni}(0, 1)$.

6. Find the integral

$$\int_0^\infty e^{-\frac{1}{2}x^2} dx$$

7. Let $X_1, X_2, \ldots$ be i.i.d. random variables with finite mean μ and variance σ^2. According to the central limit theorem, as n goes to infinity the distribution of

$$Z_n = \frac{S_n - n\mu}{\sqrt{n\sigma^2}} \text{ where } S_n = X_1 + \cdots + X_n$$

approaches that of standard normal $Z \sim \mathcal{N}(0, 1)$ with CDF $\Phi(z) \triangleq P(Z \leq z)$. Assume $X_1 \sim \textbf{Uni}(-1, 1)$. Estimate the probability $P(S_{100} > 10)$ in terms of $\Phi(\cdot)$.

8. Consider $\boldsymbol{X} \sim \textbf{PoiPro}(\lambda)$ starting at $t = 0$ with kth arrival time Y_k. Find

$$P(0 < Y_1 < 1 \text{ and } 1 < Y_2 < 2 \,|\, \text{exactly 2 arrivals in } (0, 2))$$

9. (20 points) A CD player plays music CD in random mode. The process begins with the playing of Track 1 and ends when Track 1 is about to be played for the second time (but not played). In between, the next track to play is selected uniformly random from the entire track list. Suppose a CD with 10 tracks is inserted to the CD player.

   (a) Find the probability that Track 5 is played. (hint: one-step analysis)

   (b) Find the expected number of tracks *not* played. (hint: linearity of expectation)

10. Find the moment generating function (MGF) of the second arrival time Y_2 of a Poisson process with rate λ starting at $t = 0$. (hint: memoryless property)