Final Exam

- Usage of electronic device is forbidden.
- Answers without due explanation/reasoning will not be graded.
- Each problem is 10 points unless stated otherwise.
- When solving a problem, outline your approach (4 points) followed by computation (6 points).
- Reference

$$X \sim \textbf{Ber}(p) \;\Rightarrow\; p_X(x) = p^x(1-p)^{1-x} \text{ for } x = 0, 1$$

$$X \sim \textbf{Bin}(n,p) \;\Rightarrow\; p_X(x) = \binom{n}{x} p^x(1-p)^{n-x} \text{ for } x = 0, 1, \ldots, n$$

$$X \sim \textbf{Geo}(p) \;\Rightarrow\; p_X(x) = p(1-p)^{x-1} \text{ for } x = 1, 2, \ldots$$

$$X \sim \textbf{Uni}[a,b] \;\Rightarrow\; p_X(x) = \frac{1}{b-a+1} \text{ for } x = a, a+1, \ldots, b$$

$$X \sim \textbf{Poi}(\lambda) \;\Rightarrow\; p_X(x) = e^{-\lambda}\frac{\lambda^x}{x!} \text{ for } x = 0, 1, \ldots$$

$$X \sim \textbf{Uni}(a,b) \;\Rightarrow\; f_X(x) = \frac{1}{b-a} \text{ for } a < x < b$$

$$X \sim \textbf{Exp}(\lambda) \;\Rightarrow\; f_X(x) = \lambda e^{-\lambda x} \text{ for } x > 0$$

$$X \sim \mathcal{N}(\mu,\sigma^2) \;\Rightarrow\; f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \text{ for } x \in \mathbb{R}$$

$$\text{MGF: } M_X(t) = \mathbf{E}\left(e^{tX}\right)$$

This space is designated for calculation. This space is designated for calculation. This space is designated for calculation.

1. Draw 2 cards in sequence without replacement from a deck of 52 cards. Let A be the event that the first drawn card is a red-suit card (i.e. Hearts or Diamonds), and B be the event that the second drawn card is a black-suit card (i.e. Spades or Clubs). Find $P(A|B)$ and $P(B|A)$.

2. Let $T_1 \sim \mathbf{Exp}(1)$ and $T_2 \sim \mathbf{Exp}(2)$ be independent. Compute the probability
$$P\left(1 < \frac{T_2}{T_1} < 2\right)$$

3. Let $X \sim \mathcal{N}(0, 1)$ and $Y|X = x \sim \mathcal{N}(0, |x|)$. Find the variance of Y via total variance or otherwise.

4. Consider 1D symmetric random walk moving to left or right with equal probability. Starting from the origin (Point 0), let Z be the number of times point 10 is visited before the walk returns to the origin. Find the PMF of Z.

5. Consider a class of 100 students divided into 50 pairs. Suppose each student answers a question correctly with probability 0.7. Let X be the number of students that answer the question correctly, and N be the number of pairs that both students answer the question correctly. Decide $E(N|X)$.

6. Consider a fair coin and a biased coin, which lands Heads with probability $2/3$. Suppose a coin is selected (with equal probability) and flipped twice, landing Heads both times. What is the probability that the next flip of this coin is Head?

7. Evaluate the integral
$$\int_0^1 x^3(1-x)^7 dx$$
via Bayes' Billiards argument or otherwise.

8. 3 persons take a tour bus with 5 passenger seats. Suppose the seating is uniformly random for the trip to destination and the trip back. What is the probability that at least one person takes the same seat in both trips?

9. Route-1 bus departs from terminal regularly every *half* hour. Route-2 bus departs from terminal with random exponential time between departures, with 2 departures per hour on average. Paul arrives at the terminal at uniform random time. What is the probability that he sees route-1 bus departs before route-2 bus, given that the next route-2 bus departs within an hour?

10. Let $X \sim \text{Exp}(1)$ and $Y \sim \text{Exp}(1)$ be independent. Find the expectation of
$$M = \max(X, Y)$$