Midterm on 2025 April 16

- Usage of electronic device is forbidden.
- Answers without due explanation/reasoning will not be graded.
- For your reference

$$X \sim \textbf{Ber}(p) \;\Rightarrow\; p_X(x) = p^x(1-p)^{1-x} \text{ for } x = 0, 1$$
$$X \sim \textbf{Bin}(n, p) \;\Rightarrow\; p_X(x) = \binom{n}{x} p^x(1-p)^{n-x} \text{ for } x = 0, 1, \ldots, n$$
$$X \sim \textbf{Geo}(p) \;\Rightarrow\; p_X(x) = p(1-p)^{x-1} \text{ for } x = 1, 2, \ldots$$
$$X \sim \textbf{Uni}[a, b] \;\Rightarrow\; p_X(x) = \frac{1}{b-a+1} \text{ for } x = a, a+1, \ldots, b$$
$$X \sim \textbf{Poi}(\lambda) \;\Rightarrow\; p_X(x) = e^{-\lambda}\frac{\lambda^x}{x!} \text{ for } x = 0, 1, \ldots$$

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1. Let M be the number of matches to play until the winner of a best-of-five series is decided. Assume every match is 50-50. Compute $\mathbf{E}M$ and $\operatorname{var} M$. (hint: find the distribution of M first)

2. Let $X \sim \textbf{Bin}(10, 0.3)$ and $Y \sim \textbf{Bin}(10, 0.7)$ be independent, and $Z = X + Y$. Compute $\operatorname{var} Z$ and $\operatorname{cov}(X, Z)$.

3. Consider a ring of 12 points indexed by $1, \ldots, 12$ in clockwise order (as in a clock). A random walk starts with Point 12 and ends with the first revisit back to Point 12. During a walk, the walker moves 1 step clockwise with probability 0.5 or 1 step counter-clockwise with probability 0.5. Let K be the number of visits to Point 6.
   - Compute $p_K(0)$ (hint: gambler's ruin, total probability)
   - Compute $\mathbf{E}K$ and $\operatorname{var} K$ (hint: geometric, conditional expectation)

4. Let N be the number of different birthdays for 30 classmates, where each birthday is random uniformly.
   - Find $\mathbf{E}N$ (hint: linearity of expectation)
   - Approximate $p_N(30)$ (hint: the rule of rare events)

5. In a fantasy probability class, the number of register students is Poisson $N \sim \text{Poi}(50)$. Each student passes with probability 0.9 or fails with probability 0.1, independent of each other. Define $Z = X - Y$ where X is be the number of students that pass and Y is the number of students that fail. Compute $\operatorname{var} Z$ and $\operatorname{cov}(X, Z)$. (hint: chicken-egg problem)

6. (bonus 10 points) Suppose the number of customer arrivals at campus barber shop in an interval τ is $H \sim \text{Poi}(\lambda\tau)$, where λ is 3 customers per hour. Professor C wants a haircut. He calls the barber from his office to make sure there are no customers. Suppose it takes 5 minutes for him to go to the shop. Assume each customer before him makes him wait for 10 more minutes.
   - Decide the expected time he waits at the shop to begin his haircut.
   - Suppose he has a class in 25 minutes. Compute the probability that he has his haircut and returns for the class in time.