Problem Set

1. (split arrival processes and random sum) Define

$$S = X_1 + \cdots + X_N$$

where N is a non-negative integer random variable and $X_1, X_2, \ldots$ are i.i.d. random variables. For the following cases

$$\begin{aligned} N &\sim \textbf{Poi}(\lambda), & X_1 &\sim \textbf{Ber}(q) \\ N &\sim \textbf{Geo}(q), & X_1 &\sim \textbf{Exp}(\lambda) \\ N &\sim \textbf{Geo}(q), & X_1 &\sim \textbf{Geo}(p) \\ N &\sim \textbf{Bin}(k, p), & X_1 &\sim \textbf{Ber}(q) \end{aligned}$$

   (a) identify S as arrival count/time in a Poisson/Bernoulli process $\boldsymbol{X}$ to be split

   (b) determine the distribution of S using an arrival process $\boldsymbol{Z}$ split from $\boldsymbol{X}$

2. (Bayesian learning) In estimating the parameter of a Bernoulli/Poisson process, one can apply Bayesian learning with conjugate priors.

   (a) Assume $\boldsymbol{X}$ is a Poisson process. Treat the arrival rate of $\boldsymbol{X}$ as a random variable with prior distribution $R \sim \text{Gamma}(a, \lambda)$. Update the distribution of R after observing k arrivals in an interval of length τ.

   (b) Assume $\boldsymbol{Y}$ is a Bernoulli process. Treat the probability of arrival of $\boldsymbol{Y}$ in a time slot as a random variable with prior distribution $P \sim \text{Beta}(a, b)$. Update the distribution of P after observing k arrivals in n time slots.

3. (Poisson process) Let $\boldsymbol{X} \sim \textbf{PoiPro}(\lambda)$ and $N(\tau)$ be the arrival counts in $(0, \tau)$. Show that

$$N(s) \,|\, N(t) = k \sim \textbf{Bin}\left(k, \frac{s}{t}\right), \; t > s > 0$$

4. Consider a ring of 6 points indexed by $0, \ldots, 5$ in clockwise order. A walk starts with Point 0 and ends with the first revisit of Point 0. During a walk, the walker moves $1 \ldots 6$ steps clockwise on the points of a fair dice.

   (a) Find $\mathbf{E}(T)$ where T is the number of steps for a walk to end.

   (b) Find P_1, P_2, P_3 where P_i is the probability that Point i is visited during a walk.