Homework 7

- Due by 2025.05.21 9:11AM
- Answers without due explanation/reasoning will not be graded.

1. (random sum) Define
$$S = X_1 + \cdots + X_N$$
where N is a non-negative integer random variable and $X_1, X_2, \ldots$ are i.i.d. random variables. Determine the distribution of S using MGF for the following cases.
    - $X_1 \sim \textbf{Ber}(q)$ and $N \sim \textbf{Bin}(n, p)$
    - $X_1 \sim \textbf{Geo}(q)$ and $N \sim \textbf{Geo}(p)$
2. (joint distribution) Let $\Lambda \sim \textbf{Uni}(0, 1)$ and $X|\Lambda = \lambda \sim \textbf{Ber}(\lambda)$. Find
    - the joint distribution $f_{X\Lambda}$ of X and Λ
    - the marginal PMF $p_X(0)$ and $p_X(1)$
    - the conditional PDF of $f_{\Lambda|X}(\lambda|1)$ and $f_{\Lambda|X}(\lambda|0)$
3. (central limit theorem) Let $X_1, X_2, \ldots$ be i.i.d. random variables with finite mean μ and variance σ^2. According to the central limit theorem, the distribution of
$$Z_n = \frac{S_n - n\mu}{\sqrt{n\sigma^2}} \text{ where } S_n = X_1 + \cdots + X_n$$
approaches the standard normal distribution as n increases. Estimate probability for the following events.
    - (cargo) Assume $X_1 \sim \textbf{Uni}(0, 6)$ and the event is $S_{100} > 330$.
    - (poll) Assume $X_1 \sim \textbf{Ber}(p)$ and the event is $\left|\frac{S_{1000}}{1000} - p\right| > 0.03$.
4. (random walk) Consider a ring of 7 points indexed by $0, \ldots, 6$ in clockwise order. A random walk starts with Point 0 and ends with the first revisit of Point 0. During a walk, the walker moves 1 step clockwise with probability 0.5 or 1 step counter-clockwise with probability 0.5.
    - Let T be the number of steps for a walk to end. Find $\mathbf{E}(T)$.
    - Let K_i be the number of visits to Point i during a walk. Find $\mathbf{E}(K_i), i = 1, 2, 3$.