Markov Chains

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Probability

Outline

- Discrete-time Markov chain
- Classification of states
- Steady-state behavior
- Birth-death process
- Absorption
- Continuous-time Markov chain

Discrete-time Markov Chains

Definition (Markov process)

A random process $\boldsymbol{X}$ is a Markov process if it satisfies the Markov property

$$\boldsymbol{X}_{<t} \perp\!\!\!\perp \boldsymbol{X}_{>t} \mid X_t$$

- Future and past are independent given current state
- The influence of history on future is summarized by status quo

Definition (Discrete-time Markov chain)

Let $\boldsymbol{X}$ be a Markov process.

- $\boldsymbol{X}$ is a Markov chain if the random variables of $\boldsymbol{X}$ are discrete
- $\boldsymbol{X}$ is a discrete-time Markov chain (DTMC) if the time index set is discrete

Terminology

Let $\boldsymbol{X}$ be a discrete-time Markov chain starting from $t = 0$.

- state variables: random variables denoted by $X_0, X_1, \dots$
- state visit: X_t taking a value, often represented by $X_t = i$
- state transition: can be denoted by $X_t = i \cap X_{t+1} = j$
- state space: value set of X_t, e.g. $\{0, 1, \dots, m\}$ or $\{1, \dots, m\}$

Note we assume a finite state space unless stated otherwise.

Example (Discrete-time Markov chain)

- daily stock (buy, sell, hold)
- daily weather (rain, cloudy, sunny)
- yearly snowfall (inches)
- monthly sales (up, down, flat)

Probabilities

Let $\boldsymbol{X}$ be DTMC starting from $t = 0$.

- The probabilistic distribution of $\boldsymbol{X}$ can be specified by initial state probabilities

$$P(X_0 = i)$$

  and state transition probabilities

$$P(X_{n+1} = j \,|\, X_n = i)$$

- Time homogeneity is assumed for state transition

$$P(X_{n+1} = j \,|\, X_n = i) = P(X_1 = j \,|\, X_0 = i), \ \forall\, n \geq 0$$

State transition graph

A DTMC can be represented by state transition graph.

Let $\boldsymbol{X}$ have m states. In the state transition graph, there are

- m nodes for the states of $\boldsymbol{X}$
- $n \leq (m \times m)$ directed edges for the state transitions
- n probabilities for the state transitions

Transition probability matrix

A DTMC can be represented by transition probability matrix (TPM).

Let $\boldsymbol{X}$ have m states with state transition probabilities

$$p_{ij} = P(X_{n+1} = j \mid X_n = i),\ 1 \leq i, j \leq m$$

The TPM is

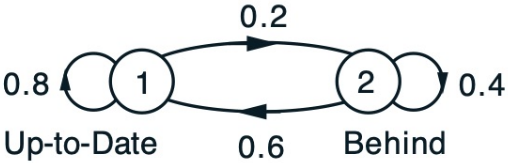
$$\boldsymbol{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix}$$

Example (7.1)

Alice takes a probability course.

- In each week, she is either **up-to-date** or **behind**
- If she is up-to-date in a given week, the probability that she will be up-to-date in the next week is 0.8
- If she is behind in a given week, the probability that she will be up-to-date in the next week is 0.6

This is a DTMC with $m = 2$ states. The representation by a state transition graph is as follows.

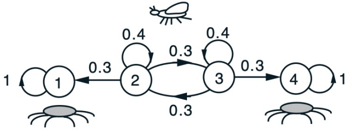


Example (7.2)

A fly moves along a straight line in unit increments.

- During each time slot, it moves one unit to the left with probability 0.3, one unit to the right with probability 0.3, and stays in place with probability 0.4.
- Two spiders are lurking at positions 1 and m. If the fly lands in these positions, it is captured by a spider.

This is a DTMC with m states. For $m = 4$, the representation by a state transition graph or a TPM is as follows.



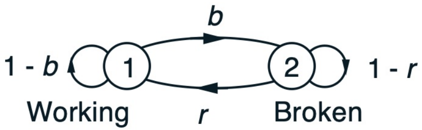
	1	2	3	4
1	1.0	0	0	0
2	0.3	0.4	0.3	0
3	0	0.3	0.4	0.3
4	0	0	0	1.0

p_{ij}

Example (7.3)

A machine is either **working** or **broken** on a given day. If it is working, it will be broken the next day with probability b, or continue to be working with probability $1 - b$. If it is broken on a given day, it will be repaired and be working on the next day with probability r, or continue to be broken with probability $1 - r$.

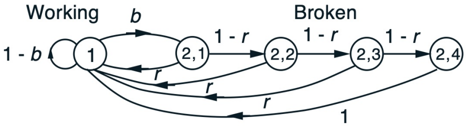
This is a DTMC with $m = 2$ states. The representation by a state transition graph is as follows.



Example (7.3 continued)

Suppose whenever the machine is broken for l consecutive days, it is replaced with a new machine.

This is a DTMC with $m = l + 1$ states. The representation by a state transition graph is as follows.



Storing the number of consecutive broken days requires l states.

Definition (State sequence)

Let $\boldsymbol{X}$ be DTMC starting from $t = 0$. A state sequence $\boldsymbol{s} = s_0 \cdots s_n$ of $\boldsymbol{X}$ is the event

$$(X_0 = s_0) \cap \cdots \cap (X_n = s_n)$$

The probability of a state sequence is

$$\begin{aligned}
&P(\boldsymbol{s}) = P(X_0 = s_0 \cap \cdots \cap X_n = s_n) \\
&= P(X_0 = s_0)P(X_1 = s_1 | X_0 = s_0)P(X_2 = s_2 | X_1 = s_1 \cap X_0 = s_0) \cdots \\
&= P(X_0 = s_0)P(X_1 = s_1 | X_0 = s_0)P(X_2 = s_2 | X_1 = s_1) \cdots \\
&= P(X_0 = s_0) \left[\prod_{k=1}^{n} p_{s_{k-1} s_k} \right]
\end{aligned}$$

Definition (n-step state transition probability)

Let $\boldsymbol{X}$ be DTMC with m states. The n-step state transition probability of $\boldsymbol{X}$ is the probability of transition from one state to another state in n steps. That is

$$r_{ij}(n) = P(X_n = j \mid X_0 = i)$$

Note

$$r_{ij}(1) = P(X_1 = j \mid X_0 = i) = p_{ij}$$

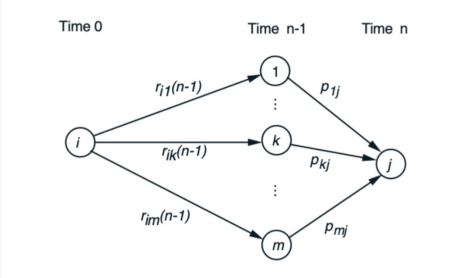
Chapman-Kolmogorov equation

Let $\boldsymbol{X}$ be DTMC with m states, p_{ij}'s be the state transition probabilities, and $r_{ij}(n)$'s be the n-step state transition probabilities.

- We have

$$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1)p_{kj}$$

- They are Chapman-Kolmogorov equations (abbr. CK Eqs)



Proof.

For $n \geq 2$

$$
\begin{aligned}
r_{ij}(n) &= P(X_n = j | X_0 = i) \\
&= \sum_{k=1}^{m} P(X_n = j \cap X_{n-1} = k | X_0 = i) \\
&= \sum_{k=1}^{m} P(X_{n-1} = k | X_0 = i) P(X_n = j | X_{n-1} = k \cap X_0 = i) \\
&= \sum_{k=1}^{m} P(X_{n-1} = k | X_0 = i) P(X_n = j | X_{n-1} = k) \\
&= \sum_{k=1}^{m} r_{ik}(n-1) p_{kj}
\end{aligned}
$$

□

n-STEP TRANSITION PROBABILITY MATRIX

Let $\boldsymbol{X}$ be DTMC with TPM $\boldsymbol{P} = \{p_{ij}\}$. Define n-step transition probability matrices $\boldsymbol{R}_n = \{r_{ij}(n)\}$. Then

$$\boldsymbol{R}_n = \boldsymbol{P}^n$$

By definition $\boldsymbol{R}_1 = \boldsymbol{P}$. For $n \geq 2$, the CK Eqs

$$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1) p_{kj}$$

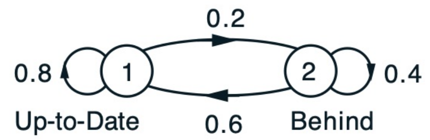
can be expressed as a matrix equation

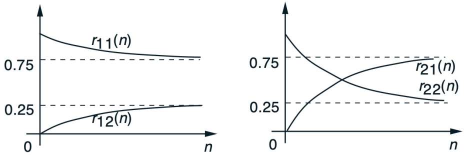
$$\boldsymbol{R}_n = \boldsymbol{R}_{n-1} \boldsymbol{P}$$

Applying the recurrence relation repeatedly, we have

$$\boldsymbol{R}_n = \boldsymbol{R}_{n-1} \boldsymbol{P} = \boldsymbol{R}_{n-2} \boldsymbol{P}^2 = \cdots = \boldsymbol{R}_1 \boldsymbol{P}^{n-1} = \boldsymbol{P}^n$$

Example (7.1)





n-step transition probabilities as a function of the number *n* of transitions

$r_{ij}(1)$

	UpD	B
UpD	0.8	0.2
B	0.6	0.4

$r_{ij}(2)$

.76	.24
.72	.28

$r_{ij}(3)$

.752	.248
.744	.256

$r_{ij}(4)$

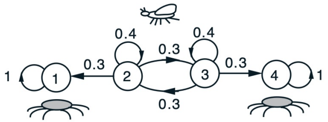
.7504	.2496
.7488	.2512

$r_{ij}(5)$

.7501	.2499
.7498	.2502

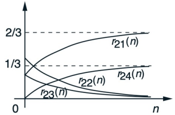
Sequence of *n*-step transition probability matrices

Example (7.2)



	1	2	3	4
1	1.0	0	0	0
2	0.3	0.4	0.3	0
3	0	0.3	0.4	0.3
4	0	0	0	1.0

p_{ij}



n-step transition probabilities as a function of the time *n*

	1	2	3	4
1	1.0	0	0	0
2	0.3	0.4	0.3	0
3	0	0.3	0.4	0.3
4	0	0	0	1.0

$r_{ij}(1)$

1.0	0	0	0
.42	.25	.24	.09
.09	.24	.25	.42
0	0	0	1.0

$r_{ij}(2)$

1.0	0	0	0
.50	.17	.17	.16
.16	.17	.17	.50
0	0	0	1.0

$r_{ij}(3)$

1.0	0	0	0
.55	.12	.12	.21
.21	.12	.12	.55
0	0	0	1.0

$r_{ij}(4)$

. . . .

1.0	0	0	0
2/3	0	0	1/3
1/3	0	0	2/3
0	0	0	1.0

$r_{ij}(\infty)$

Sequence of transition probability matrices

Classification of States

Definition (Accessible state)

Let $\boldsymbol{X}$ be DTMC.

- State j is accessible from state i if there exists a non-zero n-step transition probability from i to j, i.e.

$$\exists n,\ r_{ij}(n) = P(X_n = j | X_0 = i) > 0$$

- This is denoted by

$$i \ \rightarrow \ j$$

- $i \rightarrow j$ implies existence of a path from node i to node j in the state transition graph of $\boldsymbol{X}$ with edges pointing along the path

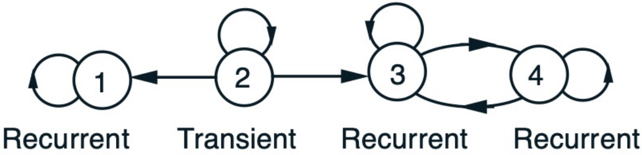
Definition (Recurrent state and transient state)

Let $\boldsymbol{X}$ be DTMC and i be a state of $\boldsymbol{X}$.

- State i is recurrent if i is accessible from any state of $\boldsymbol{X}$ that is accessible from i
- Otherwise, i is transient

Is state i recurrent? transient?

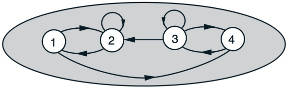
- Find the set of states $\mathcal{S}(i)$ that are accessible from i
- For each state $j \in \mathcal{S}(i)$, decide if i is accessible from j
  - i is recurrent if i is accessible from j for every $j \in \mathcal{S}(i)$
  - i is transient if $\exists\, j \in \mathcal{S}(i)$ such that i is not accessible from j



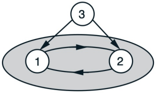
Definition (Recurrent class)

Let X be DTMC. A recurrent class of X is a set of recurrent states of X that are accessible to/from each other.

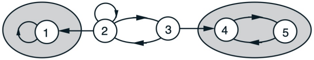
- A DTMC with a finite state space has at least one recurrent class.
- A DTMC may have multiple recurrent classes.
- Recurrent states in different recurrent classes are **not accessible** from one another.



Single class of recurrent states



Single class of recurrent states (1 and 2)
and one transient state (3)



Two classes of recurrent states
(class of state1 and class of states 4 and 5)
and two transient states (2 and 3)

Periodic/aperiodic classes

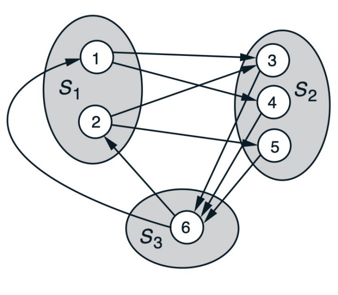
Let $\boldsymbol{X}$ be DTMC and $\mathcal{C}$ be a recurrent class of $\boldsymbol{X}$.

- $\mathcal{C}$ can be partitioned into subset(s)

$$\mathcal{C} = S_1 \cap \cdots \cap S_d$$

  where a state transition of $\mathcal{C}$ is from S_k to S_{k+1} or from S_d to S_1.

- $\mathcal{C}$ is **periodic** for $d \geq 2$. $\mathcal{C}$ is **aperiodic** for $d = 1$.
- $\mathcal{C}$ is **single** if it is the only recurrent class.



Steady-State Behavior

Definition (Steady-state probability)

Let $\boldsymbol{X}$ be DTMC with a finite state space, initial probability $\boldsymbol{p}_0$ and TPM $\boldsymbol{P}$. If $P(X_n = j)$ converges in the long run, then

$$\pi_j = \lim_{n\to\infty} P(X_n = j)$$

is the steady-state probability of state j.

Steady-state and n-step transition probability

Let $\boldsymbol{X}$ be DTMC with a finite state space and a single aperiodic recurrent class.

- Every state of $\boldsymbol{X}$ has a steady-state probability
- The n-step transition probability from state i to state j converges to the steady-state probability of j

$$\lim_{n\to\infty} r_{ij}(n) = \pi_j$$

Balance equations

Let $\boldsymbol{X}$ be DTMC with a finite state space, a single aperiodic recurrent class, and TPM $\boldsymbol{P}$.

- The steady-state probabilities satisfy

$$\pi_j = \sum_{k=1}^{m} \pi_k p_{kj}, \ j = 1, \cdots, m$$

- They are the balance equations

Recall CK Eqs for n-step transition probability

$$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1) p_{kj}$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} r_{ij}(n) = \lim_{n\to\infty} \sum_{k=1}^{m} r_{ik}(n-1) p_{kj} \ \Rightarrow \ \pi_j = \sum_{k=1}^{m} \pi_k p_{kj}$$

Frequency of state visit

Let $\boldsymbol{X}$ be DTMC with a finite state space and a single aperiodic recurrent class. The expected relative frequency of visits of state j converges to the steady-state probability π_j.

Let $I_j(t)$ be the indicator of $\boldsymbol{X}$ visiting j at time t. Let $V_j(n)$ be the count of $\boldsymbol{X}$ visiting j from time 1 to n. We have

$$V_j(n) = \sum_{t=1}^{n} I_j(t)$$

Since $\lim_{t\to\infty} P(I_j(t) = 1) = \pi_j$, we have

$$\mathbf{E}\left[\frac{V_j(n)}{n}\right] = \mathbf{E}\left[\frac{\sum\limits_{t=1}^{n} I_j(t)}{n}\right] \xrightarrow{n\to\infty} \pi_j$$

Frequency of state transition

Let $\boldsymbol{X}$ be DTMC with a finite state space and a single aperiodic recurrent class. The expected relative frequency of state transition from state j to state k converges to $\pi_j p_{jk}$.

Let $I_{j\to k}(t)$ be the indicator of $\boldsymbol{X}$ making a transition from j to k at time t. Then

$$\begin{aligned}\lim_{t\to\infty} P(I_{j\to k}(t)=1) &= \lim_{t\to\infty} P(X_t=j \cap X_{t+1}=k) \\ &= \lim_{t\to\infty} P(X_t=j)P(X_{t+1}=k|X_t=j) \\ &= \pi_j p_{jk}\end{aligned}$$

Let $Q_{jk}(n)$ be the count of $\boldsymbol{X}$ making that transition from time 1 to n. By a similar argument, we have

$$\mathbf{E}\left[\frac{Q_{jk}(n)}{n}\right] = \mathbf{E}\left[\frac{\sum_{t=1}^{n} I_{j\to k}(t)}{n}\right] \xrightarrow{n\to\infty} \pi_j p_{jk}$$

Balance equations

Consider the equations for steady-state probabilities

$$\pi_j = \sum_{k=1}^{m} \pi_k p_{kj}$$

- Steady-state probabilities must "balance" incoming and outgoing state transitions.

$$\pi_j = \pi_j \sum_{k=1}^{m} p_{jk} = \overbrace{\sum_{k=1}^{m} \pi_j p_{jk} = \sum_{k=1}^{m} \pi_k p_{kj}}^{\text{outgoing} \,=\, \text{incoming}}$$

- Define row vector $\boldsymbol{\pi} = \{\pi_j\}$ and TPM $\boldsymbol{P} = \{p_{ij}\}$.

$$\boldsymbol{\pi} = \boldsymbol{\pi}\boldsymbol{P}$$

Normalization condition

Let $\boldsymbol{X}$ be DTMC with a single aperiodic recurrent class. The steady-state probabilities of $\boldsymbol{X}$ satisfy the normalization condition

$$\sum_j \pi_j = 1$$

For any n

$$\sum_j r_{ij}(n) = \sum_j P(X_n = j | X_0 = i) = 1$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} \left(\sum_j r_{ij}(n) \right) = 1 = \sum_j \left(\lim_{n\to\infty} r_{ij}(n) \right) = \sum_j \pi_j$$

Example (7.5)

Consider a DTMC with $m = 2$ states and state transition probabilities

$$p_{11} = 0.8, \quad p_{12} = 0.2$$
$$p_{21} = 0.6, \quad p_{22} = 0.4$$

Find the steady-state probabilities of the states.

By balance equations

$$\pi_1 = \pi_1 p_{11} + \pi_2 p_{21} = 0.8\pi_1 + 0.6\pi_2$$
$$\pi_2 = \pi_1 p_{12} + \pi_2 p_{22} = 0.2\pi_1 + 0.4\pi_2$$

and normalization condition

$$\pi_1 + \pi_2 = 1$$

we can solve π_1 and π_2.

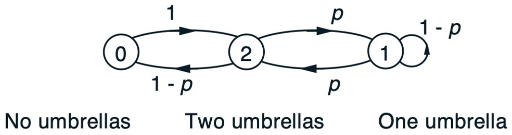
Example (7.6)

An absent-minded professor has 2 umbrellas that he uses when commuting between home to office. Suppose that it rains with probability p each time he commutes. What is the probability that he gets wet during a commute?

We have a DTMC with the following state transition graph. The steady-state probabilities satisfy balance equations

$$\pi_0 = \pi_2(1-p)$$
$$\pi_1 = \pi_1(1-p) + \pi_2 p$$
$$\pi_2 = \pi_0 \cdot 1 + \pi_1 p$$

and normalization $\pi_0 + \pi_1 + \pi_2 = 1$. The wet probability is $p\pi_0$.

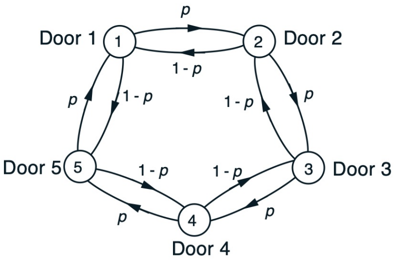


Example (7.7)

A superstitious professor works in a circular building with m doors, where m is odd, and never uses the same door twice in a row. Instead, he uses the door left with probability p or right with probability $1-p$. What is the probability that a particular door will be used one day far into the future?

Let state i represent using door i. By symmetry

$$\pi_i = \frac{1}{m}$$



Probability distribution over states

Let $\boldsymbol{X}$ be DTMC with a finite state space and a single aperiodic recurrent class. Let $\boldsymbol{P}$ be the TPM of $\boldsymbol{X}$ and row vector $\boldsymbol{p}_n = \{P(X_n = j)\}$ be the probability distribution of X_n over the states at time n.

- At time $n+1$

$$\begin{aligned} P(X_{n+1} = j) &= \sum_{i=1}^{m} P(X_{n+1} = j, X_n = i) \\ &= \sum_{i=1}^{m} P(X_n = i) P(X_{n+1} = j | X_n = i) \end{aligned}$$

- That is

$$\boldsymbol{p}_{n+1} = \boldsymbol{p}_n \boldsymbol{P}$$

- It follows that

$$\boldsymbol{p}_n = \boldsymbol{p}_0 \boldsymbol{P}^n$$

Stationarity of steady-state probabilities

Let $\boldsymbol{X}$ be DTMC with a finite state space, a single aperiodic recurrent class, TPM $\boldsymbol{P}$, and steady-state probability $\boldsymbol{\pi}$. Then $\boldsymbol{\pi}$ is stationary.

Suppose the probability distribution of X_n over the states is $\boldsymbol{\pi}$. At time $n+1$, the probability distribution of X_{n+1} over the states is

$$\boldsymbol{p}_{n+1} = \boldsymbol{p}_n \boldsymbol{P} = \boldsymbol{\pi} \boldsymbol{P} = \boldsymbol{\pi}$$

Birth-Death Processes

Definition (Birth-death processes)

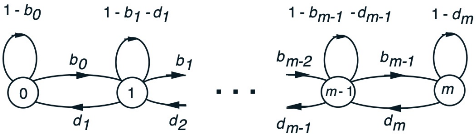
Let $\boldsymbol{X}$ be DTMC with a linear state transition graph and allow only self-transitions or transitions between neighboring states. Then $\boldsymbol{X}$ is a birth-death process.

- Transition to right neighbor is birth. The probability of birth from state i is

$$P(X_{n+1} = i + 1 \mid X_n = i) = b_i$$

- Transition to left neighbor is death. The probability of death from state i is

$$P(X_{n+1} = i - 1 \mid X_n = i) = d_i$$



Birth-death local balance

A birth-death process has local balance equations. Each local balance equation involves only 2 states.

$$\pi_{i-1} b_{i-1} = \pi_i d_i$$

The numbers of transitions from state i to state $i-1$ and from state $i-1$ to state i differ by no more than 1.

$$Q_{i-1,i}(n) - 1 \leq Q_{i,i-1}(n) \leq Q_{i-1,i}(n) + 1$$

So

$$\frac{Q_{i-1,i}(n)}{n} - \frac{1}{n} \leq \frac{Q_{i,i-1}(n)}{n} \leq \frac{Q_{i-1,i}(n)}{n} + \frac{1}{n}$$

Letting $n \to \infty$, we get $\pi_{i-1}\, p_{i-1\,i} = \pi_i\, p_{i\,i-1}$. That is

$$\pi_{i-1} b_{i-1} = \pi_i d_i$$

Birth-death steady-state probability

Let $\boldsymbol{X}$ be a birth-death process with m states, birth probability b_i's, and death probability d_i's.

- The balance equations of $\boldsymbol{X}$ are

$$\pi_1 b_1 = \pi_2 d_2, \ \cdots \ , \ \pi_{m-1} b_{m-1} = \pi_m d_m$$

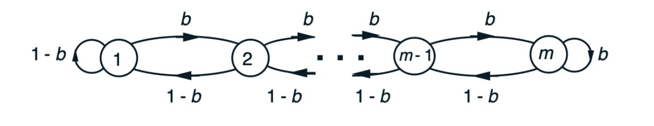
- The steady-state probabilities are related to π_1 by

$$\pi_2 = \pi_1 \frac{b_1}{d_2}, \ \cdots \ , \ \pi_m = \pi_1 \frac{b_1 \cdots b_{m-1}}{d_2 \cdots d_m}$$

- We determine π_1 by the normalization condition
- Then we find $\pi_2, \cdots, \pi_m$

Example (7.8)

Yoko walks along a straight line and, at each time slot, takes a step to the right with probability b, and a step to the left with probability $1-b$. She starts in one of the positions $1, \cdots, m$, but if she reaches the position 0 (or position $m+1$), her step is instantly reflected back to 1 (or m, respectively). Introduce a DTMC whose states are the positions $1, \cdots, m$. What are the steady-state probabilities?

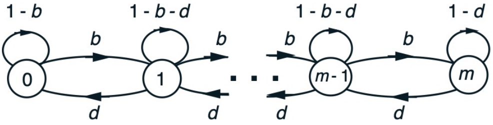


Example (7.9)

Packets arrive at a node of a communication network, where they are stored in a buffer of size m and then transmitted. If m packets are already in the buffer, any newly arriving packets are discarded. We discretize time into very small slots, and assume that in each slot, exactly one of the following events occurs.

- One new packet is stored to the buffer, with probability $b > 0$ if the buffer is not full
- One packet stored in the buffer completes transmission, with probability $d > 0$ if the buffer is non-empty
- No new packet is stored and no stored packet completes transmission, with probability $1 - b - d$ if the buffer is non-empty, and with probability $1 - b$ if the buffer is empty

Introduce a DTMC whose states are the number of packets in the buffer $0, 1, \cdots, m$. What are the steady-state probabilities?



The DTMC is a birth-death process, with

$$b_0 = \cdots = b_{m-1} = b, \quad b_m = 0$$
$$d_0 = 0, \quad d_1 = \cdots = d_m = d$$

The local balance equations are $\pi_{i-1} b = \pi_i d$. Thus

$$\pi_i = \left(\frac{b}{d}\right) \pi_{i-1} = \cdots = \left(\frac{b}{d}\right)^i \pi_0, \; i = 1, \cdots, m$$

π_0 can be found by normalization

$$\pi_0 = \frac{1}{1 + \rho + \cdots + \rho^m}, \; \rho = \frac{b}{d}$$

and $\pi_i = \rho^i \pi_0$.

Absorption

Absorbing state and absorption probability

Let $\boldsymbol{X}$ be DTMC with a finite state space.

- A state of $\boldsymbol{X}$ is an absorbing state if it does not transit to any other states
- Let s be an absorbing state. We have

$$p_{ss} = 1$$

- The absorption probability of s from state i is defined by

$$a_i = P\left(\lim_{n\to\infty} X_n = s \;\middle|\; X_0 = i\right)$$

Absorption probability equation

Let $\boldsymbol{X}$ be DTMC with a finite state space. Consider the absorption probabilities of absorbing state s.

- If state i is recurrent

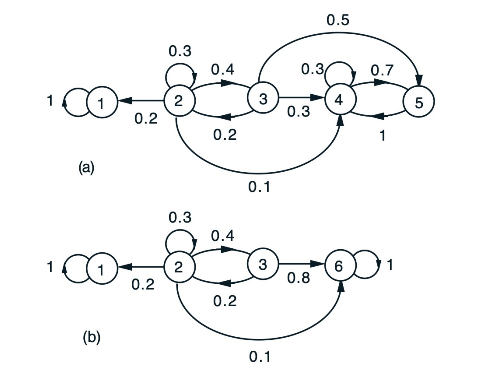
$$a_i = \delta_{is}$$

- If state i is transient

$$\begin{aligned} a_i &= P\left(\lim_{n\to\infty} X_n = s | X_0 = i\right) \\ &= \sum_{j=1}^{m} P\left(\lim_{n\to\infty} X_n = s \cap X_1 = j | X_0 = i\right) \\ &= \sum_{j=1}^{m} P\left(X_1 = j | X_0 = i\right) P\left(\lim_{n\to\infty} X_n = s | X_1 = j\right) \\ &= \sum_{j=1}^{m} p_{ij} a_j \end{aligned}$$

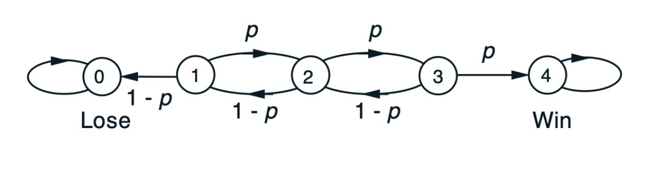
Example (7.10)

DTMC (a) has 2 recurrent classes, namely $\{1\}$ and $\{4, 5\}$. Merging states $4, 5$ as a single absorbing state 6, we have DTMC (b). The probability that DTMC (a) enters the recurrent class $\{4, 5\}$ from a state is an absorption probability of state 6 in DTMC (b).



Example (7.11)

A gambler wins \$1 at each round with probability p, and loses \$1 with probability $1-p$. He gambles until he either accumulates \$$m$ or loses all his money. What is the probability that he wins?



Probability of winning depends on the amount of money he has. Let state i represent the state that the gambler has i dollars. The probability that he wins is the absorption probability of m. For the recurrent states

$$a_m = 1, \; a_0 = 0$$

For the transient states

$$a_i = \sum_{j=0}^{m} p_{ij} a_j = (1-p)a_{i-1} + p a_{i+1}, \quad i = 1, \cdots, m-1$$

$$\Rightarrow \ (1-p)(a_i - a_{i-1}) = p(a_{i+1} - a_i)$$

$$\Rightarrow \ (a_{i+1} - a_i) = \left(\frac{1-p}{p}\right)(a_i - a_{i-1})$$

$$\Rightarrow \ \delta_i = \rho\delta_{i-1}, \text{ where } \delta_i = a_{i+1} - a_i, \ \rho = \frac{1-p}{p}$$

$$\Rightarrow \ \delta_i = \rho\delta_{i-1} = \cdots = \rho^i \delta_0$$

From $a_m - a_0 = (a_m - a_{m-1}) + \cdots + (a_1 - a_0)$, we have

$$1 - 0 = \delta_{m-1} + \cdots + \delta_0 = \delta_0(\rho^{m-1} + \cdots + 1)$$

$$\Rightarrow \ \delta_0 = \frac{1}{\rho^{m-1} + \cdots + 1}$$

$$\Rightarrow \ a_i = a_0 + \delta_0 + \cdots + \delta_{i-1} = \frac{\rho^{i-1} + \cdots + \rho^0}{\rho^{m-1} + \cdots + 1}$$

Definition (Time to absorption)

Let $\boldsymbol{X}$ be DTMC with a finite state space.

- The time to absorption is the first time to a recurrent state

$$T = \min\{n \geq 0, X_n = j \text{ where } j \text{ is recurrent}\}$$

- The expected time to absorption from state i is

$$\mu_i = \mathbf{E}[T|X_0 = i]$$

Change of mode. Entering a recurrent state marks the switch of a DTMC from short-term mode to long-term mode.

Expected time to absorption

Let $\boldsymbol{X}$ be DTMC and μ_i be expected time to absorption of state i.

$$\mu_i = \begin{cases} 0, & \text{if } i \text{ is recurrent} \\ 1 + \sum_{j=1}^{m} p_{ij}\mu_j, & \text{if } i \text{ is transient} \end{cases}$$

$$\begin{aligned} \mu_i &= \mathbf{E}[T \,|\, X_0 = i] = \mathbf{E}[\mathbf{E}[T|X_1] \,|\, X_0 = i] \\ &= \sum_{j=1}^{m} P(X_1 = j \,|\, X_0 = i)\, \mathbf{E}[T \,|\, X_1 = j \cap X_0 = i] \\ &= \sum_{j=1}^{m} p_{ij}\mathbf{E}[T \,|\, X_1 = j] = \sum_{j=1}^{m} p_{ij}(1 + \mathbf{E}[T \,|\, X_0 = j]) \\ &= \sum_{j=1}^{m} p_{ij}(\mu_j + 1) = 1 + \sum_{j=1}^{m} p_{ij}\mu_j \end{aligned}$$

Example (Poe divination)

For fair wood pieces, what is the expected number of drops until back-to-back divines occur?

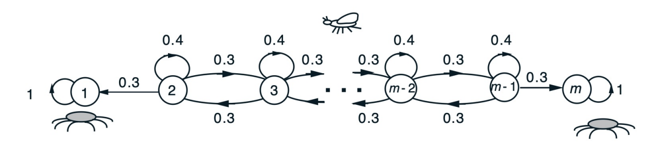
Define 3 states, where state i means needing i consecutive Ds to end the drops. State 0 is an absorbing state. We have

$$\begin{aligned} \mu_0 &= 0 \\ \mu_1 &= 1 + 0.5\mu_0 + 0.5\mu_2 \\ \mu_2 &= 1 + 0.5\mu_2 + 0.5\mu_1 \end{aligned}$$

The solution is $\mu_1 = 4$ and $\mu_2 = 6$. Note $\mu_2 = 6$ agrees with the conclusion reached earlier via total expectation.

Example (7.12)

Consider Example 7.2 with $m = 4$. What is the expected time for the fly to be captured by a spider?



States 1 and 4 are recurrent states. We have

$$\begin{aligned} \mu_1 &= 0 \\ \mu_2 &= 1 + 0.3\mu_1 + 0.4\mu_2 + 0.3\mu_3 \\ \mu_3 &= 1 + 0.3\mu_2 + 0.4\mu_3 + 0.3\mu_4 \\ \mu_4 &= 0 \\ \Rightarrow \quad \mu_2 &= \mu_3 = \frac{10}{3} \end{aligned}$$

Continuous-time Markov Chains

Definition (Continuous-time Markov chain)

A continuous-time Markov chain (abbr. CTMC)

- is a continuous-time random process
- satisfies Markov property
- has a discrete state space

We assume time homogeneity in our discussion of CTMC.

CTMC variables

Let $\boldsymbol{X}$ be a CTMC.

- The random variables of $\boldsymbol{X}$ are

$$X_0 \xrightarrow{T_1} X_1 \xrightarrow{T_2} X_2 \xrightarrow{T_3} \cdots \xrightarrow{T_n} X_n \xrightarrow{T_{n+1}} X_{n+1} \cdots$$

- X_n is the state variable after the nth state transition, not the state variable at time n
- T_n is the waiting time (in state X_{n-1}) of the nth state transition

CTMC probability models

Let $\boldsymbol{X}$ be a CTMC with a finite state space.

- The waiting time for state transition is an exponential random variable

$$T_{n+1} \,|\, (X_n = i) \sim \textbf{Exp}(\nu_i)$$

- The distribution of next state is independent of the waiting time

$$P(X_{n+1} = j \,|\, X_n = i, T_n) = P(X_{n+1} = j \,|\, X_n = i) = p_{ij}$$

$$P(X_{n+1} = i \,|\, X_n = i) = p_{ii} = 0$$

CTMC embedding

Let $\boldsymbol{X}$ be a CTMC with a finite state space. Consider the sequence of state variables

$$\boldsymbol{X}' : X_0 \, X_1 \, \cdots$$

- $\boldsymbol{X}'$ is an embedded DTMC of $\boldsymbol{X}$
- The TPM of $\boldsymbol{X}'$ is

$$\boldsymbol{P} = \{p_{ij}\}$$

Exiting rate and transition rate

Let $\boldsymbol{X}$ be a CTMC.

- The expected time to transition from state i is

$$\mathbf{E}[T_{n+1} \mid X_n = i] = \int_0^\infty t\nu_i e^{-\nu_i t} dt = \frac{1}{\nu_i}$$

- It takes ν_i^{-1} to exit state i on average, so there are ν_i exits per unit time
- The state exiting rate from state i is ν_i
- A proportion p_{ij} of the state transitions out of state i go to state j
- The state transition rate from state i to state j is

$$q_{ij} = \nu_i p_{ij}$$

Example (7.14)

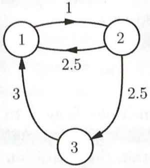
A machine, once in **production** mode, operates continuously until an alarm signal is generated. The time up to the alarm signal is an exponential random variable with parameter 1. Subsequent to the alarm signal, the machine is in **test** mode for an exponentially distributed amount of time with parameter 5. The test results are positive, with probability $1/2$, in which case the machine returns to production mode, or negative, with probability $1/2$, in which case the machine is taken for **repair**. The duration of the repair is exponentially distributed with parameter 3. Construct a CTMC for this machine.

The states are **production**, **test**, and **repair**. The state transition probabilities are

$$\boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix}$$

The state exiting rates are $\nu_1 = 1, \nu_2 = 5, \nu_3 = 3$. So the state transition rates are

$$\boldsymbol{Q} = \{\nu_i p_{ij}\} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{5}{2} & 0 & \frac{5}{2} \\ 3 & 0 & 0 \end{bmatrix}$$



CTMC characterization

Let $\boldsymbol{X}$ be a CTMC. State exiting rates and state transition probabilities of $\boldsymbol{X}$ can be derived from the state transition rates of $\boldsymbol{X}$.

Suppose the state transition rates of $\boldsymbol{X}$ are

$$q_{ij}, \quad i \neq j$$

- The state exiting rates of $\boldsymbol{X}$ are

$$\nu_i = \nu_i \sum_j p_{ij} = \sum_j \nu_i p_{ij} = \sum_j q_{ij}$$

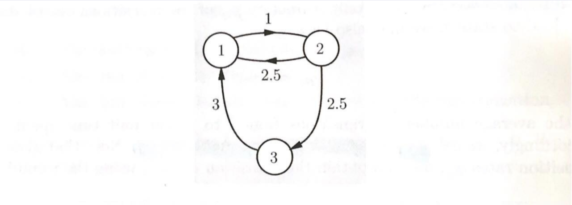
- The state transition probabilities of $\boldsymbol{X}$ are

$$p_{ij} = \frac{q_{ij}}{\nu_i} = \frac{q_{ij}}{\sum_{j'} q_{ij'}}$$

CTMC graph representation

A CTMC can be represented by a graph.

- States are represented by nodes
- State transitions are represented by directed edges, which are labelled by the state transition rates



From such a graph, we can determine model parameters μ_i and p_{ij}.

Auxiliary DTMC

Let $\boldsymbol{X}$ be a CTMC and $X(t)$ be the state variable at time t. Let $\boldsymbol{Z}$ be the discrete-time random process defined by $Z_n = X(n\delta)$ where $\delta > 0$.

- $\boldsymbol{Z}$ satisfies the Markov property $\boldsymbol{Z}_{<n} \perp\!\!\!\perp \boldsymbol{Z}_{>n} \,|\, Z_n$
- It is an auxiliary DTMC of $\boldsymbol{X}$

Let $\bar{\boldsymbol{P}} = \{\bar{p}_{ij}\}$ be the TPM of $\boldsymbol{Z}$. For $j \neq i$

$$\begin{aligned}\bar{p}_{ij} &= P(Z_{n+1} = j \,|\, Z_n = i) \\ &= \underbrace{(\nu_i \delta + o(\delta))}_{\text{transition from } i \text{ in } \delta} \times \underbrace{p_{ij}}_{\text{transition to } j} \\ &= \nu_i \, p_{ij} \, \delta + o(\delta) \\ &= q_{ij} \, \delta + o(\delta)\end{aligned}$$

For $j = i$

$$\bar{p}_{ii} = 1 - \sum_{j \neq i} \bar{p}_{ij} = 1 - \delta \sum_{j \neq i} q_{ij} + o(\delta)$$

Example (7.14 continued)

Let $\boldsymbol{X}$ be a CTMC for the machine. Let $\boldsymbol{Z}$ be an auxiliary DTMC of $\boldsymbol{X}$ with $Z_n = X(n\delta)$. If we neglect $o(\delta)$ terms, the TPM of $\boldsymbol{Z}$ is

$$\bar{\boldsymbol{P}} = \begin{bmatrix} 1-\delta & \delta & 0 \\ \frac{5}{2}\delta & 1-5\delta & \frac{5}{2}\delta \\ 3\delta & 0 & 1-3\delta \end{bmatrix}$$

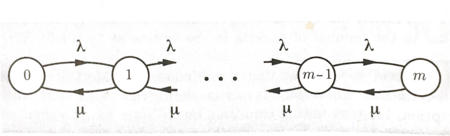
CTMC alternative characterization

Let $\boldsymbol{X}$ be a continuous-time random process with a finite value set (state space). $\boldsymbol{X}$ is a CTMC if there exist $q_{ij} \geq 0$ such that

$$P(X(t+\delta) = j | X(t) = i) = \begin{cases} q_{ij}\delta + o(\delta), & j \neq i \\ 1 - \sum_{j \neq i} q_{ij}\delta + o(\delta), & j = i \end{cases}$$

Example (7.15)

Let the arrivals of packets at a node be a Poisson process with rate λ. Upon arrival, a packet is either stored in a buffer for m packets, or discarded if the buffer is full. The time to transmit a packet is exponential with parameter μ. Show that the buffer can be modeled as a CTMC.



Let $\boldsymbol{X}$ be a random process counting the packets in the buffer. $\boldsymbol{X}$ is a CTMC as we have

$$P(X(t+\delta) = i-1 \,|\, X(t) = i) = \mu\delta + o(\delta), \quad i = 1, \cdots, m$$
$$P(X(t+\delta) = i+1 \,|\, X(t) = i) = \lambda\delta + o(\delta), \quad i = 0, \cdots, m-1$$

CTMC steady-state probabilities

Let $\boldsymbol{X}$ be a CTMC with a finite state space.

- The steady-state probability of state j is defined by

$$\pi_j = \lim_{t \to \infty} P(X(t) = j)$$

- Let $\boldsymbol{Z}$ be an auxiliary DTMC of $\boldsymbol{X}$. Then

$$\lim_{t \to \infty} P(X(t) = j) = \lim_{n \to \infty} P(Z_n = j)$$

CTMC balance equations

Let $\boldsymbol{X}$ be a CTMC with a finite state space, a single recurrent class, and state transition rates q_{ij}. The steady-state probabilities of $\boldsymbol{X}$ satisfy the balance equations

$$\pi_j \sum_{k \neq j} q_{jk} = \sum_{k \neq j} \pi_k q_{kj}, \ \forall\, j$$

Let $\boldsymbol{Z}$ be an auxiliary DTMC of $\boldsymbol{X}$ with $Z_n = X(n\delta)$. We have

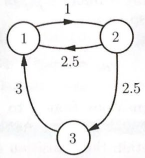
$$\begin{aligned} \pi_j &= \sum_k \pi_k \bar{p}_{kj} = \pi_j \bar{p}_{jj} + \sum_{k \neq j} \pi_k \bar{p}_{kj} \\ &= \pi_j \left(1 - \delta \sum_{k \neq j} q_{jk} + o(\delta) \right) + \sum_{k \neq j} \pi_k \left(q_{kj}\delta + o(\delta) \right) \end{aligned}$$

Thus

$$\pi_j \sum_{k \neq j} q_{jk} = \sum_{k \neq j} \pi_k q_{kj}$$

Example (7.14 continued)

What are the steady-state probabilities of Example 7.14?



The balance equations are

$$\pi_1(1+0) = \frac{5}{2}\pi_2 + 3\pi_3, \pi_2\left(\frac{5}{2}+\frac{5}{2}\right) = \pi_1 + 0\pi_3, \pi_3(3+0) = 0\pi_1 + \frac{5}{2}\pi_2$$

We also have

$$\pi_1 + \pi_2 + \pi_3 = 1$$

Hence

$$\pi_1 = \frac{30}{41},\ \pi_2 = \frac{6}{41},\ \pi_3 = \frac{5}{41}$$

Continuous-time birth-death processes

A continuous-time birth-death process is a CTMC, in which the states are linearly arranged and only transitions to neighboring states are possible.

Local balance equations

For a continuous-time birth-death process, the transition rate from i to $i+1$ equals the transition rate from $i+1$ to i.

$$\pi_i q_{i\,i+1} = \pi_{i+1} q_{i+1\,i}$$

Example (7.15 continued)

What are the steady-state probabilities of Example 7.15?

The local balance equations are

$$\pi_i \lambda = \pi_{i+1}\mu \;\Rightarrow\; \pi_{i+1} = \pi_i \left(\frac{\lambda}{\mu}\right) = \pi_i \rho$$
$$\Rightarrow\; \pi_i = \pi_0 \rho^i$$

We find π_0 by normalization equation $\pi_0 + \pi_1 + \cdots + \pi_m = 1$.

$$\pi_0(1 + \rho + \cdots + \rho^m) = 1 \;\Rightarrow\; \pi_0 = (1 + \rho + \cdots + \rho^m)^{-1}$$

Then we find the other probabilities

$$\pi_i = \frac{\rho^i}{1 + \rho + \cdots + \rho^m}, \; \rho = \frac{\lambda}{\mu}$$

Summary 1

Chapman-Kolmogorov equations for n-step transition

$$r_{ij}(1) = p_{ij},\ r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1)p_{kj} \quad (\boldsymbol{R}_n = \boldsymbol{P}^n)$$

Steady-state convergence theorem

$$\lim_{n\to\infty} r_{ij}(n) = \lim_{n\to\infty} P(X_n = j)$$

Balance equations of DTMC

$$\pi_j = \sum_{k=1}^{m} \pi_k p_{kj},\ j = 1, \cdots, m$$

Exponential waiting time to a state transition

$$f_{T_{n+1}|X_n=i}(t) = \nu_i e^{-\nu_i t} u(t-0)$$

State transition probability

$$P(X_{n+1}=j|X_n=i) = p_{ij}$$

Transition rate to state j while in state i

$$q_{ij} = \nu_i p_{ij}$$

An auxiliary DTMC $\boldsymbol{Z}$ of a CTMC $\boldsymbol{X}$

$$Z_n = X(n\delta)$$

Balance equations of CTMC

$$\pi_j \sum_{k \neq j} q_{jk} = \sum_{k \neq j} \pi_k q_{kj}$$

Local balance equations of CTMC

$$\pi_i q_{ij} = \pi_j q_{ji}, \; j = i \pm 1$$